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par

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Variétés de représentations de carquois à boucles

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Résumé

Cette thèse s'articule autour des espaces de modules de représentations de carquois arbitraires, c'est-à-dire possédant d'éventuelles boucles. Nous obtenons trois types de résultats. Le premier concerne la base canonique de Lusztig, dont la définition est étendue à notre cadre, notamment en introduisant une algèbre de Hopf généralisant les groupes quantiques usuels (*i.e.* associés aux algèbres de Kac-Moody symétriques). On démontre au passage une conjecture faite par Lusztig en 1993, portant sur la catégorie de faisceaux pervers qu'il définit sur les variétés de représentations de carquois.

Le second type de résultats, également inspiré par le travail de Lusztig, concerne la base semi-canonique et la variété Lagrangienne nilpotent de Lusztig. Pour un carquois arbitraire, on définit des sous-variétés de représentations *semi-nilpotentes* $\Lambda(\alpha)$, et nous montrons qu'elles sont Lagrangiennes. La démonstration repose sur l'existence de fibrations affines partielles entre diverses composantes de $\Lambda(\alpha)$, contrôlées par une combinatoire précise. Nous définissons une algèbre de convolution de fonctions constructibles sur $\sqcup\Lambda(\alpha)$, et montrons qu'elle possède une base formée de fonctions *quasi-caractéristiques* des composantes irréductibles des $\Lambda(\alpha)$. La structure combinatoire qui se dégage ici est analogue à celle obtenue sur les faisceaux pervers de Lusztig, et fait apparaître des opérateurs plus généraux que ceux décrits par les cristaux de Kashiwara.

Le troisième thème considéré est celui des variétés carquois de Nakajima, dont l'étude géométrique menée ici permet, conjointement avec ce qui est fait précédemment, de donner une définition de cristaux de Kashiwara généralisés. On définit à nouveau des sous-variétés Lagrangiennes, ainsi qu'un produit tensoriel sur leurs composantes irréductibles, comme fait dans le cas classique par Nakajima.

Mots-clefs : Carquois à boucles, Base canonique, Base semi-canonique, Variétés carquois de Nakajima, Cristaux de Kashiwara.

VARIETIES OF REPRESENTATIONS OF QUIVERS WITH LOOPS

Abstract

This thesis is about the moduli spaces of representations of arbitrary quivers, *i.e.* possibly carrying loops. We obtain three types of results. The first one deals with the Lusztig canonical basis, whose definition is here extended to our framework, thanks in particular to the definition of a Hopf algebra generalizing the usual quantum groups (*i.e.* associated to symmetric Kac-Moody algebras). We also prove a conjecture raised by Lusztig in 1993, which concerns the category of perverse sheaves he defines on varieties of representations of quivers.

The second type of results, also inspired by the work of Lusztig, concerns the semicanonical basis. For an arbitrary quiver, we define subvarieties of *seminilpotent* representations $\Lambda(\alpha)$, and we show that they are Lagrangian. The proof relies on the existence of partial affine fibrations between some irreducible components of $\Lambda(\alpha)$, controled by a precise combinatorial structure. We define a convolution algebra of constructible functions on $\sqcup\Lambda(\alpha)$, and show it is equipped with a basis of *quasi-characteristic* functions of the irreducible components of the $\Lambda(\alpha)$. The combinatorial structure arising from this construction is analogous to the one obtained on Lusztig perverse sheaves, and yields operators more general than the ones described by Kashiwara crystals.

The third considered topic is the one of Nakajima quiver varieties, whose geometric study in this thesis allows, along with the previous (also geometric) work, to define generalized Kashiwara crystals. We define, again, Lagrangian subvarieties, and a tensor product of their irreducible components, as done by Nakajima on the classical case.

Keywords : Quivers with loops, Canonical basis, Semicanonical basis, Nakajima quiver varieties, Kashiwara Crystals.

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Table des matières

0 Introduction

0.1 Contexte

Cette thèse s'inscrit dans le cadre de la théorie géométrique des représentations, qui consiste à donner des interprétations géométriques à des objets de théorie de représentations définis *a priori* algébriquement, et ainsi en donner des propriétés non trivivales, notamment via l'étude d'espaces de modules de représentations.

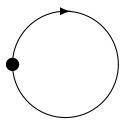
Ici nous allons nous intéresser aux représentations de carquois, *i.e.* de graphes orientés $Q = (I, \Omega)$ donnés par un ensemble de sommets I et un ensemble de flèches $\Omega = \{h : s(h) \to t(h)\}$ entre sommets. Une représentation d'un carquois Q sur une corps k consiste alors en la donnée d'une famille de k-espaces vectoriels $V = (V_i)_{i \in I}$ indexée par I, et une famille d'applications linéaires :

$$(x_h)_{h \in \Omega} \in E_V = \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{s(h)}, V_{t(h)})$$

indéxée par Ω. Les représentations d'un carquois sur un corps forment une catégorie abélienne k-linéaire. Ces représentations ont gagné une grande importance dans les années 70, notamment via le théorème de Gabriel :

Théorème 0.1 (Gabriel, 1972)**.** *Les carquois possédant une nombre fini d'isoclasses de représentations indécomposables sont ceux dont le graphe sous-jacent est un diagramme de Dynkin simplement lacé, i.e. de type A_n, D_n, E_6, E_7 ou E_8. Les isoclasses de représentations indécomposables sont alors en correspondance bijective avec les racines positives du système de racines du diagramme de Dynkin concerné.*

Ce théorème fait ainsi le lien avec la classification des algèbres de Lie semi-simples complexes de dimension finie. Parallèlement, le cas du carquois de Jordan $\tilde{A}_0$:



est aussi bien connu : la classification de ses isoclasses de représentations est (tautologiquement) équivalente à celle des classes de conjugaison des matrices à coefficients dans k. En particulier, si k est algébriquement clos, on dispose de la réduction de Jordan et on a une correspondance bijective entre les isoclasses de représentations nilpotentes en dimension n et les partitions de n.

Dans la suite on va voir comment le théorème de Gabriel peut être d'une part amélioré, en utilisant plus en détails la géométrie de variétés de repésentations de carquois pour en tirer des propriétés algébriques en théorie de Lie, et d'autre part étendu, mettant en rapport une plus large classe d'algèbres de Lie avec des carquois plus généraux.

0.2 Bases canoniques et semi-canoniques de Lusztig

Lusztig définit dans une série d'articles deux types de bases de la partie positive de l'algèbre enveloppante $\mathbf{U}^+(\mathfrak{g})$, issues de la géométrie des variétés de représentations de carquois, et pour des carquois de plus en plus généraux.

La première, dite *base canonique*, fait intervenir la théorie des faisceaux pervers développée dans [BBD82]. Pour la définir, il réalise la partie positive $\mathbf{U}_v^+(\mathfrak{g})$ du groupe quantique comme le groupe de Grothendieck associé à une certaine classe de faisceaux pervers. Elle est définie dans [Lus90] pour les carquois de type fini, puis dans [Lus91] (et dans [Kas91], *c.f.* 0.3) pour les carquois sans boucles de type "infini".

La seconde, dite *semi-canonique*, fait elle intervenir des fonctions constructibles sur le champ cotangent au champ de modules de représentations du carquois Q, obtenue cette fois seulement pour le paramètre quantique $v = 1$. Elle est d'abord définie dans [Lus92] dans le cas des carquois de type affine (ou fini), puis généralisée au craquois sans boucles dans [Lus00], mais à l'aide d'un résultat obtenu dans [KS97] grâce à la théorie des cristaux (*c.f.* 0.3).

En particulier dans toute cette section, les carquois considérés sont sans boucles.

0.2.1 Base canonique

Définissons la classe de faisceaux pervers qui nous intéresse. Pour tout vecteur dimension $\alpha = \sum_{i\in I} \alpha_i i \in \mathbb{N}I$, on commence par fixer un espace vectoriel I-gradué V_α de dimension α, et on note $E_\alpha = E_{V_\alpha}$. Soient alors $\mathbf{i} = (i_1, \dots, i_m)$ et $\mathbf{a} = (a_1, \dots, a_m)$ deux suites finies d'éléments de I et $\mathbb{N}_{>0}$ respectivement, telles que $\sum_{1\le k\le m} a_k i_k = \alpha$. On pose :

$$\mathcal{F}_{\mathbf{i},\mathbf{a}} = \left\{ \mathsf{W} = (\{0\} = \mathsf{W}_0 \subset \dots \subset \mathsf{W}_m = V_\alpha) \,\middle|\, \forall k, \dim \frac{\mathsf{W}_k}{\mathsf{W}_{k-1}} = a_k i_k \right\}$$
$$\widetilde{E}_{\mathbf{i},\mathbf{a}} = \{(x, \mathsf{W}) \mid x_h(\mathsf{W}) \subseteq \mathsf{W}\} \subseteq E_\alpha \times \mathcal{F}_{\mathbf{i},\mathbf{a}}.$$

La première projection fournit un morphisme propre $\pi_{\mathbf{i},\mathbf{a}} : \widetilde{E}_{\mathbf{i},\mathbf{a}} \to E_\alpha$. D'après le théorème de décomposition de Beilinson, Bernstein et Deligne, le complexe $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ est semi-simple ($\mathbf{1}$ désigne le faisceau constant sur $\widetilde{E}_{\mathbf{i},\mathbf{a}}$. On note alors :

- ▷ G_α le groupe $\prod_{i\in I} GL((V_\alpha)_i)$, qui agit naturellement sur E_α ;
- ▷ $\mathcal{M}_{G_\alpha}(E_\alpha)$ la catégorie des faisceaux pervers G_α-équivariants sur E_α ;
- ▷ $\mathcal{P}_\alpha$ la sous-catégorie pleine de $\mathcal{M}_{G_\alpha}(E_\alpha)$ consistant en les sommes de faisceaux pervers simples G_α-équivariants apparaissant, éventuellement décalés, comme facteurs directs de $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ pour une certaine paire $(\mathbf{i}, \mathbf{a})$ telle que $\sum a_k i_i = \alpha$;
- ▷ $\mathcal{Q}_\alpha$ la categorie des complexes isomorphes à des sommes de décalages d'objets de $\mathcal{P}_\alpha$;
- ▷ $\mathcal{K}_\alpha$ le groupe de Grothendieck de $\mathcal{Q}_\alpha$, vu comme un $\mathbb{Z}[v^{\pm 1}]$-module en posant $v^{\pm 1}[\mathbf{P}] = [\mathbf{P}[\pm 1]]$, où l'on note $[\mathbf{P}]$ l'isoclasse d'un faisceau pervers $\mathbf{P}$;
- ▷ $\mathcal{B}_\alpha$ l'ensemble fini des isoclasses des faisceaux simples de $\mathcal{P}_\alpha$ et $\mathcal{B} = \sqcup_\alpha \mathcal{B}$.

Définissons maintenant des foncteurs de *restriction* et d'*induction* qui permettront de munir $\mathcal{K} = \oplus_\alpha \mathcal{K}_\alpha$ d'une structure d'algèbre de Hopf.

Pour tout sous-espace I-gradué $W \subseteq V_\alpha$ de dimension β et codimension γ, muni de deux isomorphismes I-gradués $p : W \xrightarrow{\sim} V_\beta$ and $q : V_\alpha/W \xrightarrow{\sim} V_\gamma$, on obtient le diagramme suivant :

$$E_\beta \times E_\gamma \xleftarrow{\kappa} E_\alpha(W) \xrightarrow{\iota} E_\alpha$$

où $E_\alpha(W) = \{x \in E_\alpha \mid x(W) \subseteq W\}$, κ désigne le fibré vectoriel $x \mapsto (p_*(x_W), q_*(x_{V_\alpha/W}))$ et ι l'inclusion.

On considère aussi :

$$E_\beta \times E_\gamma \xleftarrow{p_1} E^\dagger_{\beta,\gamma} \xrightarrow{p_2} E_{\beta,\gamma} \xrightarrow{p_3} E_\alpha$$

où :

$$E^\dagger_{\beta,\gamma} = \left\{ (x, W, r, \bar{r}) \;\middle|\; \begin{array}{l} x \in E_\alpha \\ W \subseteq V_\alpha \text{ est } I\text{-gradué et } x\text{-stable} \\ r : W \tilde{\to} V_\beta \\ \bar{r} : V_\alpha/W \tilde{\to} V_\gamma \end{array} \right\}$$

$$E_{\beta,\gamma} = \left\{ (x, W) \;\middle|\; \begin{array}{l} x \in E_\alpha \\ W \subseteq V_\alpha \text{ est } I\text{-gradué et } x\text{-stable} \end{array} \right\}.$$

Ces diagrammes induisent (*cf.* [Lus10, §9.2]) :

$$\widetilde{\mathrm{Res}}_{\beta,\gamma} = \kappa_! \iota^* : \mathcal{Q}_\alpha \to \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta$$
$$\widetilde{\mathrm{Ind}}_{\beta,\gamma} = p_{3!} p_{2\flat} p_1^* : \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta \to \mathcal{Q}_\alpha$$

et :

$$\mathrm{Res}_{\beta,\gamma} = \widetilde{\mathrm{Res}}^\alpha_{\beta,\gamma}[(\gamma,\beta) - \langle \beta,\gamma\rangle]$$
$$\mathrm{Ind}_{\beta,\gamma} = \widetilde{\mathrm{Ind}}^\alpha_{\beta,\gamma}[(\gamma,\beta) + \langle \beta,\gamma\rangle]$$

où $\langle \beta,\gamma\rangle = \sum_{i\in I} \beta_i\gamma_i$ et $(\gamma,\beta) = \sum_{h\in\Omega} \gamma_{s(h)}\beta_{t(h)}$.

Rappelons aussi brièvement la définition des groupes quantiques, tout du moins celle de leur partie positive. Soit $\mathfrak{g}$ une algèbre de Kac-Moody, on fixe une décomposition de Cartan $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, et on note $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ la sous-algèbre de Borel positive associée. L'algèbre $\mathbf{U}_v(\mathfrak{b}^+)$ est engendrée par des éléments $K_i^{\pm 1}$, E_i ($i \in I$) sujets aux relations suivantes :

$$K_iK_j = K_jK_i$$
$$K_iE_j = v^{a_{i,j}} E_jK_i$$
$$\sum_{t+t'=-a_{i,j}+1} (-1)^t E_j^{(t)} E_i E_j^{(t')} = 0$$

où $a_{i,j}$ désigne l'opposé du nombre d'arêtes de Ω reliant i et j, et $E_i^{(t)} = E_i^t/[t]!$, où :

$$[t] = \frac{v^t - v^{-t}}{v - v^{-1}}.$$

On peut munir $\mathbf{U}_v(\mathfrak{b}^+)$ d'une structure d'algèbre de Hopf, le coproduit étant donné par :

$$\Delta(K_i) = K_i \otimes K_i$$
$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i.$$

On notera $\mathbf{U}_v^+(\mathfrak{g})$ la sous-algèbre engendrée par les E_i, et $\mathbf{U}_{v,\mathbb{Z}}^+(\mathfrak{g})$ sa forme intégrale, *i.e.* la $\mathbb{Z}[v^{\pm 1}]$-algèbre engendrée par les E_i construite de manière analogue.

Ces algèbres sont munies de formes de Hopf, dites *géométrique* pour $\mathcal{K}$, *de Drinfeld* pour $\mathbf{U}_v^+(\mathfrak{g})$.

On obtient alors :

Théorème 0.2 (Lusztig, 90, 91)**.** *On a un isomoprhisme d'algèbres de Hopf :*

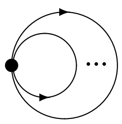
$$\begin{aligned}\Psi : \mathbf{U}_{v,\mathbb{Z}}^+(\mathfrak{g}) &\to \mathcal{K}\\ E_i^{(a)} &\mapsto [\pi_{i,a!}\mathbf{1}].\end{aligned}$$

Dans cet énoncé, les $\pi_{i,a!}\mathbf{1}$ correspondent aux plus 'simples' des faisceaux de Lusztig, obtenus quand $\mathbf{i}$ et $\mathbf{a}$ sont des suites à un élément. La base canonique est alors définie par :

$$\mathbf{B} = \{\Psi^{-1}(b) \mid b \in \mathcal{B}\}.$$

0.2.2 Une conjecture

Dans l'article [Lus93], Lusztig se demande dans quelle généralité les classes des faisceaux "monômes" $[\pi_{i,a!}\mathbf{1}]$ suffisent à engendrer le groupe de Grothendieck $\mathcal{K}$. C'est une conséquence du théorème 0.2 dans le cas des carquois sans boucles, et c'est un résultat classique dans le cas du carquois de Jordan (voir par exemple [Sch09a] et les références qui s'y trouvent). Il donne dans cet article une condition pour que cette propriété soit vraie, qui s'applique notamment aux carquois à un sommet et $g \geq 2$ boucles :



Sa conjecture est la suivante :

Conjecture 0.3. *Les classes de faisceaux pervers* $[\pi_{i,a!}\mathbf{1}]$ *(*$i \in I$*,* $a > 0$*) suffisent à engendrer* $\mathcal{K}$ *quelque soit le carquois considéré* Q*.*

0.2.3 Base semi-canonique

On commence par dédoubler le carquois Q (toujours supposé sans boucles pour l'instant), c'est-à-dire remplacer chaque flèche h de Ω par une paire de flèche de sens opposés $(h, \bar{h})$. On obtient un carquois $\bar{Q} = (I, H = \Omega \sqcup \bar{\Omega})$ qui ne dépend pas de l'orientation de Q et dont l'ensemble de flèches est stable sous l'action de l'involution $h \mapsto (\bar{h} : t(h) \to s(h))$.

On note cette fois :

$$\bar{E}_V = \bigoplus_{h \in \Omega} \operatorname{Hom}(V_{s(h)}, V_{t(h)}),$$

puis $\bar{E}_\alpha = \bar{E}_{V_\alpha}$ l'espace des représentations de $\bar{Q}$ en dimension α. Celui s'identifie au fibré cotangent de E_α, et est ainsi équipé d'une forme symplectique :

$$\omega_\alpha(x, x') = \sum_{h \in H} \operatorname{Tr}(\epsilon(h) x_h x'_{\bar{h}})$$

préservée par l'action naturelle de G_α sur $\bar{E}_\alpha$. L'application moment associée $\mu_\alpha : \bar{E}_\alpha \to \mathfrak{g}_\alpha = \oplus_{i\in I} \operatorname{End}(V_\alpha)_i$ est donnée par :

$$\mu_\alpha(x) = \sum_{h\in H} \epsilon(h) x_{\bar{h}} x_h,$$

où l'on a identifié $\mathfrak{g}_\alpha^*$ et $\mathfrak{g}_\alpha$ via la fonction trace. Le champ cotangent au champ de modules de représentations $[E_\alpha/G_\alpha]$ s'identifie à $[\mu_\alpha^{-1}(0)/G_\alpha]$.

Définition 0.4. Un élément $x \in \bar{E}_\alpha$ est dit *nilpotent* s'il existe un drapeau I-gradué $\mathsf{W} = (\mathsf{W}_0 = \{0\} \subset \ldots \subset \mathsf{W}_r = V_\alpha)$ de V_α tel que $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}$ pour tout $h \in H$. On note :

$$\Lambda(\alpha) = \{x \in \mu_\alpha^{-1}(0) \mid x \text{ nilpotent}\}.$$

Remarque 0.5. Pour l'instant les carquois sont sans boucles, donc il est équivalent de demander $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet$ dans la définition ci-dessus. Celle-ci équivaut à demander l'existence d'un rang N tel que pour tout chemin $(h_1, \ldots, h_r)$ de H de longueur $r \geq N$, on ait $x_{h_1} \circ \ldots \circ x_{h_r} = 0$.

Proposition 0.6. *La sous-variété $\Lambda(\alpha) \subseteq \bar{E}_\alpha$ est Lagrangienne.*

On note $\mathcal{M}(\alpha)$ le $\mathbb{Q}$-espace vectoriel des fonctions constructibles $\Lambda(\alpha) \to \mathbb{Q}$, constantes sur les G_α-orbites. On pose $\mathcal{M} = \oplus_{\alpha\geq 0}\mathcal{M}(\alpha)$, qui peut-être munie d'une structure d'algèbre graduée. On note 1_i la fonction qui envoie sur 1 le seul élément de $\Lambda(i)$, et $\mathcal{M}_\circ$ la sous-algèbre de $\mathcal{M}$ engendrée par ces fonctions. On a le résultat suivant :

Théorème 0.7 (Lusztig, 91)**.** *On a un isomorphisme :*

$$\begin{aligned} \mathbf{U}^+(\mathfrak{g}) &\to \mathcal{M}_\circ \\ E_i &\mapsto 1_i. \end{aligned}$$

Si de plus Z est une composante irréductible de $\Lambda(\alpha)$ et $f \in \mathcal{M}(\alpha)$, on note $\rho_Z(f) = c$ si $Z \cap f^{-1}(c)$ est un ouvert dense (*i.e.* non vide) de Z. On dispose de la proposition suivante :

Proposition 0.8. *Pour tout composante irréductible Z de $\Lambda(\alpha)$, il existe une unique fonction $f \in \mathcal{M}_\circ(\alpha)$ telle que $\rho_Z(f) = 1$ et $\rho_{Z'}(f) = 0$ si $Z' \neq Z$.*

Dans [Lus92], Lusztig prouve qu'en fait la famille libre $(f_Z)_{Z\in \operatorname{Irr}\Lambda}$, où $\operatorname{Irr}\Lambda$ désigne l'ensemble des composantes irréductibles de $\sqcup_\alpha \Lambda(\alpha)$, est une base de $\mathcal{M}_\circ$. La base semi-canonique est définie comme l'image inverse de la famille (f_Z) par l'isomorphisme $\mathbf{U}^+(\mathfrak{g}) \simeq \mathcal{M}_\circ$.

On a finalement :

$$\mathcal{B} \underset{\text{base}}{\subset} \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{K} \xleftarrow{\sim} \mathbf{U}_v^+(\mathfrak{g}) \xrightarrow{v\to 1} \mathbf{U}^+(\mathfrak{g}) \xrightarrow{\sim} \mathcal{M}_\circ \underset{\text{base}}{\supset} (f_Z)_{Z\in \operatorname{Irr}\Lambda}$$

On verra comment ce résultat se généralise à tous les carquois sans boucles dans la section 0.3.

0.3 Cristaux de Kashiwara

Les cristaux sont des objects combinatoires associés aux algèbres de Lie, et qui ont des applications dans l'étude des variétés carquois, comme suggérés plus haut. Donnons-en la définition avant de comprendre en quoi ils sont utiles dans la définition des bases canonique et semi-canonique. Les résultats exposés dans cette section sont prouvés pour les carquois sans boucles.

0.3.1 Quelques définitions et propriétés

Définition 0.9. Soit $\mathfrak{g}$ une algèbre de Kac-Moody symétrisable, et $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ une décomposition de Cartan. On note P le réseau de poids associé. On fixe deux bases duales $(\alpha_i)_{i\in I}$ (racines simples) et $(h_i)_{i\in I}$ (coracines simples) de P et P^* respectivement telles que les $\langle h_i, \alpha_j\rangle$ soient les coefficients de la matrice de Cartan associée à $\mathfrak{g}$. On appelle cristal un ensemble $\mathcal{B}$ muni d'applications :

$$\begin{aligned} \mathrm{wt} &: \mathcal{B} \to P \\ \epsilon_i &: \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\} \\ \phi_i &: \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\} \\ \tilde{e}_i, \tilde{f}_i &: \mathcal{B} \to \mathcal{B} \sqcup \{0\} \end{aligned}$$

telles que pour tous $b, b' \in \mathcal{B}$, les axiomes suivants soient vérifiés :

(A1) $\phi_i(b) = \epsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$;

(A2) si $\tilde{e}_i b \neq 0$:

$$\begin{aligned} \mathrm{wt}(\tilde{e}_i b) &= \mathrm{wt}(b) + \alpha_i \\ \epsilon_i(\tilde{e}_i b) &= \epsilon_i(b) - 1 \\ \phi_i(\tilde{e}_i b) &= \phi_i(b) + 1 \ ; \end{aligned}$$

(A3) si $\tilde{f}_i b \neq 0$:

$$\begin{aligned} \mathrm{wt}(\tilde{f}_i b) &= \mathrm{wt}(b) - \alpha_i \\ \epsilon_i(\tilde{f}_i b) &= \epsilon_i(b) + 1 \\ \phi_i(\tilde{f}_i b) &= \phi_i(b) - 1 \ ; \end{aligned}$$

(A4) $\tilde{f}_i b = b' \Leftrightarrow b = \tilde{e}_i b'$;

(A5) $\phi_i(b) = -\infty \Rightarrow \tilde{e}_i(b) = \tilde{f}_i(b) = 0$.

Example 0.10. On peut définir un cristal $\mathcal{B}_i$ pour tout $i \in I$ par :

$$\begin{aligned} &\mathcal{B}_i = \{b_i(n) \mid n \in \mathbb{Z}\} \\ &\mathrm{wt}(b_i(n)) = n\alpha_i \\ &\phi_i(b_i(n)) = n \ , \ \epsilon_i(b_i(n)) = -n \\ &\phi_j(b_i(n)) = \epsilon_j(b_i(n)) = -\infty \text{ si } j \neq i \\ &\tilde{e}_i(b_i(n)) = b_i(n+1) \ , \ \tilde{f}_i(b_i(n)) = b_i(n-1) \\ &\tilde{e}_j(b_i(n)) = \tilde{f}_j(b_i(n)) = 0 \text{ si } j \neq i. \end{aligned}$$

On notera simplement b_i au lieu de $b_i(0)$.

Il existe une notion de morphisme de cristaux :

Définition 0.11. Un morphisme de cristaux $\psi : \mathcal{B} \to \mathcal{B}'$ entre deux cristaux $\mathcal{B}$ et $\mathcal{B}'$ est une application $\mathcal{B} \sqcup \{0\} \to \mathcal{B}' \sqcup \{0\}$ vérifiant pour tous $b \in \mathcal{B}$, $i \in I$:

▷ $\psi(0) = 0$;

▷ si $\psi(b) \in \mathcal{B}'$:

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)\ ,\ \epsilon_i(\psi(b)) = \epsilon_i(b)\ ,\ \phi_i(\psi(b)) = \phi_i(b)\ ;$$

▷ si $b' = \tilde{f}_i(b)$ et $\psi(b), \psi(b') \in \mathcal{B}'$:

$$\begin{aligned}\tilde{f}_i(\psi(b)) &= \psi(\tilde{f}_i(b))\\ \tilde{e}_i(\psi(b')) &= \psi(\tilde{e}_i(b')).\end{aligned}$$

Un morphisme est par ailleurs dit *strict* s'il commute à l'action de tous les $\tilde{e}_i$ et $\tilde{f}_i$, sans restriction. Une morphisme est appelé *plongement* si l'application induite $\mathcal{B} \sqcup \{0\} \to \mathcal{B}' \sqcup \{0\}$ est injective, *isomorphisme* si elle est bijective.

Il est possible d'associer naturellement un cristal $\mathcal{B}(\infty)$ au groupe quantique $\mathbf{U}_v(\mathfrak{g})$. Ce dernier est obtenu à partir de $\mathbf{U}_v^+(\mathfrak{g})$ en rajoutant un jeu de générateurs $(F_i)_{i\in I}$ vérifiant :

$$\begin{aligned}K_iF_j &= v^{-a_{i,j}}F_jK_i\\ \sum_{t+t'=-a_{i,j}+1} (-1)^t F_j^{(t)}F_iF_j^{(t')} &= 0\end{aligned}$$

et des relations de Drinfeld, non données ici, reliant les E_i et les F_i.

Pour définir $\mathcal{B}(\infty)$, on commence par définir les *opérateurs de Kashiwara*. On peut montrer qu'il existe des opérateurs e_i' et e_i'' de $\mathbf{U}_v(\mathfrak{n}^-)$ vérifiant, pour tout $z \in \mathbf{U}_v(\mathfrak{n}^-)$:

$$[E_i, z] = \frac{K_ie_i'(z) - K_i^{-1}e_i''(z)}{v - v^{-1}}.$$

Il existe une décomposition :

$$\mathbf{U}_v(\mathfrak{n}^-) = \bigoplus_{n\geq 0} F_i^{(n)} \ker e_i'.$$

Pour $z = z_0 + F_iz_1 + \cdots + F_i^{(n)}z_n \in \mathbf{U}_v(\mathfrak{n}^-)$, on définit les opérateurs de Kashiwara ainsi :

$$\begin{aligned}\tilde{e}_i(z) &= \sum_{1\leq k\leq n} F_i^{(k-1)}z_k\\ \tilde{f}_i(z) &= \sum_{0\leq k\leq n} F_i^{(k+1)}z_k.\end{aligned}$$

Le cristal $\mathcal{B}(\infty)$ est finalement donné par :

$$\mathcal{B}(\infty) = \{\tilde{f}_{i_1}\ldots\tilde{f}_{i_r}1 \mid i_1, \ldots, i_r \in I\}.$$

On peut de manière analogue associer un cristal $\mathcal{B}(\lambda)$ à tout $\mathbf{U}_v(\mathfrak{g})$-module simple de plus haut poids $V(\lambda)$.

Une notion cruciale est celle de produit tensoriel de cristaux :

Définition 0.12. Le *produit tensoriel* $\mathcal{B} \otimes \mathcal{B}' = \{b \otimes b' \mid b \in \mathcal{B}, b' \in \mathcal{B}'\}$ de deux cristaux $\mathcal{B}$ et $\mathcal{B}'$ est défini par :

$$\begin{aligned}
\mathrm{wt}(b \otimes b') &= \mathrm{wt}(b) + \mathrm{wt}(b') \\
\epsilon_i(b \otimes b') &= \max\{\epsilon_i(b), \epsilon_i(b') - \langle e_i, \mathrm{wt}(b)\rangle\} \\
\phi_i(b \otimes b') &= \max\{\phi_i(b) + \langle e_i, \mathrm{wt}(b')\rangle, \phi_i(b')\} \\
\tilde{e}_i(b \otimes b') &= \begin{cases} \tilde{e}_i(b) \otimes b' & \text{si } \phi_i(b) \geq \epsilon_i(b') \\ b \otimes \tilde{e}_i(b') & \text{si } \phi_i(b) < \epsilon_i(b'); \end{cases} \\
\tilde{f}_i(b \otimes b') &= \begin{cases} \tilde{f}_i(b) \otimes b' & \text{si } \phi_i(b) > \epsilon_i(b') \\ b \otimes \tilde{f}_i(b') & \text{si } \phi_i(b) \leq \epsilon_i(b'); \end{cases}
\end{aligned}$$

Kashiwara et Saito obtiennent dans [KS97] la caractérisation suivante du cristal $\mathcal{B}(\infty)$:

Proposition 0.13. *Soit $\mathcal{B}$ un cristal et b_0 un élément de $\mathcal{B}$ de poids nul. Supposons :*

1. $\mathrm{wt}(\mathcal{B}) \subset -\sum_{i\in I} \mathbb{N}\alpha_i$ *;*
2. b_0 *est le seul élément de $\mathcal{B}$ de poids nul ;*
3. $\epsilon_i(b_0) = 0$ *pour tout $i \in I$;*
4. $\epsilon_i(b) \in \mathbb{Z}$ *pour tous $b \in \mathcal{B}$, $i \in I$;*
5. *pour tout $i \in I$ il existe un plongement strict $\Psi_i : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}_i$;*
6. $\Psi_i(\mathcal{B}) \subset \mathcal{B} \times \{\tilde{f}_i^n(b_i) \mid n \geq 0\}$ *;*
7. *Pour tout $b \in \mathcal{B} \setminus \{b_0\}$, il existe $i \in I$, $n > 0$ et $b' \in \mathcal{B}$ tels que $\Psi_i(b) = b' \otimes \tilde{f}_i^n(b_i)$;*

Alors $\mathcal{B}$ est isomorphe à $\mathcal{B}(\infty)$.

0.3.2 Application aux variétés carquois

Pour comprendre comment la théorie des cristaux peut être reliée à la base semi-canonique, il faut revenir à [Lus91], et à la preuve du caractère Lagrangien des variétés $\Lambda(\nu)$ ou cette fois le vecteur dimension $\nu = \sum_{i\in I} \nu_i\alpha_i$ est vu comme un élément de P. On notera $\mathrm{Irr}\,\Lambda(\nu)$ l'ensemble (fini) de leurs composantes irréductibles. Lusztig définit des stratifications de ces variétés ($i \in I$) :

$$\Lambda(\nu) = \bigsqcup_{l\geq 0} \Lambda(\nu)_{i,l}$$

où :

$$\Lambda(\nu)_{i,l} = \left\{ x \in \Lambda(\nu) \mid \operatorname{codim} \sum_{\substack{h\in H \\ t(h)=i}} \operatorname{Im} x_h = l \right\},$$

qui induisent des bijections ($i \in I$, $l > 0$) :

$$\tilde{e}_{i,l} : \mathrm{Irr}\,\Lambda(\nu)_{i,l} \overset{\sim}{\leftrightarrow} \mathrm{Irr}\,\Lambda(\nu - l\alpha_i)_{i,0} : \tilde{f}_{i,l} : .$$

Kashiwara et Saito se servent de ce jeu de bijections pour définir une structure de cristal sur $\operatorname{Irr}\Lambda$ en posant pour toute composante $Z \in \operatorname{Irr}\Lambda(\nu)_{i,l}$:

$$\begin{aligned}
\mathrm{wt}(Z) &= -\nu \\
\epsilon_i(Z) &= l \\
\tilde{e}_i(Z) &= \begin{cases} 0 & \text{si } l = 0 \\ \tilde{e}_{i,l}\tilde{f}_{i,l-1}(Z) & \text{sinon} \end{cases} \\
\tilde{f}_i(Z) &= \tilde{e}_{i,l}\tilde{f}_{i,l+1}(Z).
\end{aligned}$$

Ils montrent alors que ce cristal vérifie les hypothèses de la proposition 0.13. En particulier, par définition de $\mathcal{B}(\infty)$, on obtient l'égalité suivante :

$$\dim \mathbf{U}_v^+(\mathfrak{g})[\nu] = |\operatorname{Irr}\Lambda(\nu)|$$

qui permet à Lusztig de généraliser sa définition de la base semi-canonique à tous les carquois dans [Lus00].

0.4 Liens avec les variétés carquois de Nakajima

Les variétés varquois de Nakajima, définies au début des années 90, ont eu de nombreuses applications en théorie géométriques des représentations. Elles permettent en particulier de réaliser géométriquement de nombreux objets algébriquement, via des techniques cohomologiques ou de K-théorie. On va voir dans la suite comment elles ont notamment permis des interprétations géométriques des cristaux associés aux modules de plus haut poids, et des produits tensoriels de ces cristaux.

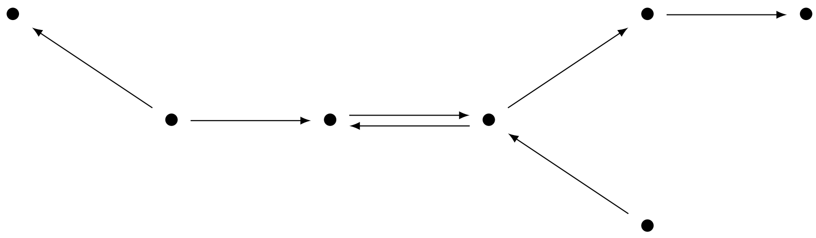
Dans la suite les réulstats sont donnés dans le cadre des algèbres de Kac-Moody symétriques.

0.4.1 Réalisation géométrique de $\mathcal{B}(\lambda)$

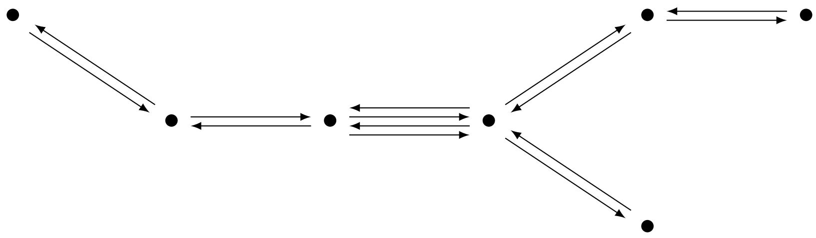
On commence par rappeler quelques définitions. Soient $(\Lambda_i)_{i\in I}$ les poids dominants, *i.e.* vérifiant $\langle h_i, \Lambda_j\rangle = \delta_{i,j}$. On commence par fixer $\mathsf{w} = \sum_{i\in I} \mathsf{w}_i\Lambda_i$ et un espace I-gradué W de dimension $(\mathsf{w}_i)_{i\in I}$. On notera $(x, f, g) = ((x_h)_{h\in H}, (f_i)_{i\in I}, (g_i)_{i\in I})$ les éléments de l'espace suivant :

$$E(V, \mathsf{w}) = \bar{E}(V, V) \oplus \bigoplus_{i\in I} \operatorname{Hom}(V_i, W_i) \bigoplus_{i\in I} \operatorname{Hom}(W_i, V_i)$$

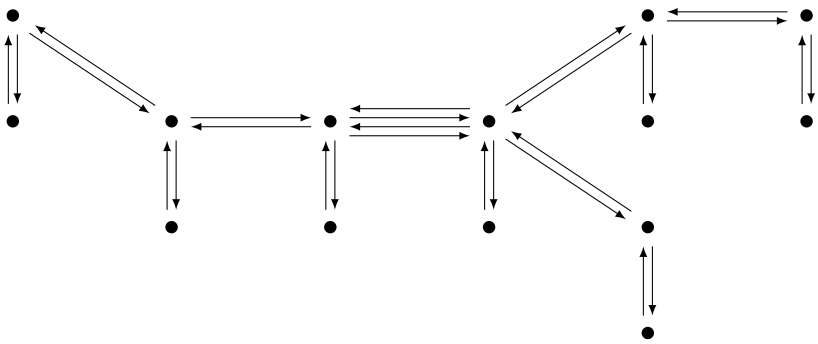
défini pour tout espace I-gradué V. Pour tout $\mathsf{v} = \sum_{i\in I} \mathsf{v}_i\alpha_i$, on pose $E_{\mathsf{v},\mathsf{w}} = E(V_{\mathsf{v}}, \mathsf{w})$. C'est l'espace de représentation en dimension (v, w) du carquois Γ obtenu à partir de $\bar{Q}$ en dédoublant les sommets et en rajoutant une paire de flèches de sens opposés entre chaque sommet de I et son double. Par exemple :

$Q =$ 

$$\Downarrow$$

$\bar{Q} =$ 

$$\Downarrow$$

$\Gamma =$ 

L'espace $E_{\mathsf{v},\mathsf{w}}$ est muni d'une forme symplectique :

$$\omega_{\mathsf{v},\mathsf{w}}\big((x,f,g),(x',f',g')\big) = \sum_{h\in H} \mathrm{Tr}(\epsilon(h)x_h x'_{\bar{h}}) + \sum_{i\in I} \mathrm{Tr}(g_i f'_i - g'_i f_i)$$

préservée par l'action naturelle de G_{v}. Cette fois, l'application moment associée $\mu_{\mathsf{v},\mathsf{w}} : E_{\mathsf{v},\mathsf{w}} \to \mathfrak{g}_{\mathsf{v}} = \oplus_{i\in I} \mathrm{End}(V_{\mathsf{v}})_i$ est donnée par :

$$\mu_{\mathsf{v},\mathsf{w}}(x,f,g) = \Big(g_i f_i + \sum_{h\in H: s(h)=i} \epsilon(h) x_{\bar{h}} x_h\Big)_{i\in I}.$$

Définition 0.14. Soit $\chi : G_{\mathsf{v}} \to \mathbb{C}^*$, $(g_i)_{i\in I} \mapsto \prod_{i\in I} \det^{-1} g_i$. On note :

$$\mathfrak{M}_\circ(\mathsf{v},\mathsf{w}) = \mu_{\mathsf{v},\mathsf{w}}^{-1}(0)/\!\!/G_{\mathsf{v}}$$
$$\mathfrak{M}(\mathsf{v},\mathsf{w}) = \mu_{\mathsf{v},\mathsf{w}}^{-1}(0)/_\chi G_{\mathsf{v}}$$

les quotients géométrique et symplectique de $\mu_{\mathsf{v},\mathsf{w}}^{-1}(0)$ par G_{v} (par rapport à χ). On note $\mathfrak{L}(\mathsf{v},\mathsf{w})$ la fibre au-dessus de 0 du morphisme projectif $\mathfrak{M}(\mathsf{v},\mathsf{w}) \to \mathfrak{M}_\circ(\mathsf{v},\mathsf{w})$.

Dans [Nak98], Nakajima définit des stratifications analogues à celles de Lusztig ($i \in I$) :

$$\mathfrak{L}(\mathsf{v},\mathsf{w}) = \bigsqcup_{l\geq 0} \mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l}$$

qui induisent à nouveau des bijections :

$$\mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l} \xrightarrow{\sim} \mathfrak{L}(\mathsf{v}-l\alpha_k,\mathsf{w})_{i,0},$$

sous la condition $l + \langle h_i, \mathsf{w}-\mathsf{v}\rangle \geq 0$, qui permettent notamment de montrer le caractère Lagrangien de $\mathfrak{L}(\mathsf{v},\mathsf{w}) \subseteq \mathfrak{M}(\mathsf{v},\mathsf{w})$. Elles permettent aussi à Saito dans l'article [Sai02] de prouver, avec des techniques analogues à celles de [KS97], le résultat suivant :

Théorème 0.15. *Il existe une structure de cristal isomorphe à* $\mathcal{B}(\mathsf{w})$ *sur les composantes irréductibles de* $\mathfrak{L}(\mathsf{w}) = \sqcup_{\mathsf{v}}\mathfrak{L}(\mathsf{v},\mathsf{w})$.

0.4.2 Réalisation géométrique du produit tensoriel de cristaux

Dans [Nak01], Nakajima construit une variété Lagrangienne $\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}') \subset \mathfrak{M}(\mathsf{v}+\mathsf{v}',\mathsf{w}+\mathsf{w}')$ telle que :

▷ il existe une bijection :

$$\operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}') \xrightarrow{\sim} \operatorname{Irr}\mathfrak{L}(\mathsf{v},\mathsf{w}) \times \operatorname{Irr}\mathfrak{L}(\mathsf{v}',\mathsf{w}') \ ; \tag{0.16}$$

▷ si l'on note $\widetilde{\mathfrak{Z}}(\mathbf{v}) = \sqcup_{\mathsf{v}+\mathsf{v}'=\mathbf{v}}\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}')$, il existe encore une stratification ($i \in I$) :

$$\widetilde{\mathfrak{Z}}(\mathbf{v}) = \bigsqcup_{l\geq 0}\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l}$$

qui induit les bijections suivantes ($i \in I$, $l > 0$) :

$$\operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l} \xrightarrow{\sim} \widetilde{\mathfrak{Z}}(\mathbf{v}-l\alpha_i)_{i,0}.$$

À nouveau, ces bijections permettent de donner une structure de cristal à $\operatorname{Irr}\widetilde{\mathfrak{Z}}$.

Le résultat suivant donne alors une interprétation géométrique au produit tensoriel de cristaux :

Théorème 0.17. *Via 0.16, le cristal* $\operatorname{Irr}\widetilde{\mathfrak{Z}}$ *est isomorphe au produit tensoriel de cristaux* $\operatorname{Irr}\mathfrak{L}(\mathsf{w}) \otimes \operatorname{Irr}\mathfrak{L}(\mathsf{w}')$.

Ce théorème, associé à un résultat de Joseph (*c.f.* [Jos95]), permet en fait de retrouver le réulstat de Saito 0.15. Avant de le donner, voyons quelques nouvelles définitions relatives aux cristaux.

Définition 0.18. Un cristal $\mathcal{B}$ est dit *normal* si pour tout $b \in \mathcal{B}$ on a :

$$\epsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k(b) \neq 0\}$$
$$\phi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k(b) \neq 0\}.$$

Définition 0.19. Un cristal $\mathcal{B}$ est dit *de plus haut poids* $\lambda \in P^+ = \sum_{i\in I}\mathbb{N}\Lambda_i$ si :

1. il existe $b_\lambda \in \mathcal{B}$ tel que $\mathrm{wt}(b_\lambda) = \lambda$ et $\tilde{e}_i(b_\lambda) = 0$ pour tout $i \in I$;
2. $\mathcal{B}$ est engendré par b_λ, *i.e.* tout $b \in \mathcal{B}$ est obtenu par applications successives des $\tilde{f}_i$.

Par exemple, les cristaux $\mathcal{B}(\lambda)$ sont des cristaux normaux de plus haut poids. Enfin :

Définition 0.20. Soit une famille $\{\mathcal{D}(\lambda) \mid \lambda \in P^+\}$ de cristaux normaux $\mathcal{D}(\lambda)$ de plus haut poids λ, contenant un élément b_λ vérifiant les conditions de la précédente définition. Cette famille est dite *fermée* si pour tous λ, μ le sous-cristal de $\mathcal{D}(\lambda) \otimes \mathcal{D}(\mu)$ engendré par $b_\lambda \otimes b_\mu$ est isomorphe à $\mathcal{D}(\lambda + \mu)$.

On peut alors donner la caractérisation suivante :

Théorème 0.21 (Joseph)**.** *Si $\{\mathcal{D}(\lambda) \mid \lambda \in P^+\}$ est une famille close de cristaux normaux de plus haut poids, alors, pour tout $\lambda \in P^+$, $\mathcal{D}(\lambda)$ est isomorphe à $\mathcal{B}(\lambda)$ en tant que cristal.*

On retrouve donc bien 0.15. En effet, d'après 0.17, la famille $\{\operatorname{Irr} \mathfrak{L}(\mathsf{w}) \mid \mathsf{w} \in P^+\}$ est fermée. Il n'est par ailleurs pas compliqué de prouver que les $\operatorname{Irr} \mathfrak{L}(\mathsf{w})$ sont normaux de plus haut poids (où b_{w} est le seul élément de $\operatorname{Irr} \mathfrak{L}(0, \mathsf{w})$), d'où le résultat.

0.5 Dans cette thèse

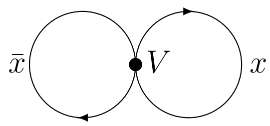
Le travail exposé dans ce manuscrit consiste à étudier la géométrie des variétés de représentations des carquois arbitraires, comportant d'éventuelles boucles, et d'en déduire une généralisation des groupes quantiques et cristaux de Kashiwara. Dans cette généralité, la matrice de Cartan est remplacée par une matrice dite de *Borcherds-Cartan*, dont les coefficients diagonaux sont égaux à :

$$c_{i,i} = 2 - 2(\text{nombre de boucles de } \Omega \text{ en } i) \in \{2, 0, -2, -4, \ldots\}.$$

Du point de vue de Lusztig, deux approches sont possibles : celle utilisant les faisceaux pervers, et celle utilisant les fonctions constructibles sur certaines sous-variétés Lagrangienne.

0.5.1 Des sous-variétés Lagrangiennes pour généraliser les cristaux

Cette thèse commence par aborder le point de vue des sous-variétés Lagrangiennes. Dans le cas général, le cas du carquois de Jordan suffit à montrer qu'il ne suffit pas de considérer des sous-variétés de $\mu^{-1}(0)$ de représentations nilpotentes, celles-ci étant trop petites. En fait, on sait dans ce cas qu'il faut considérer des représentations *semi-nilpotentes* du carquois double de $\tilde{A}_0$:

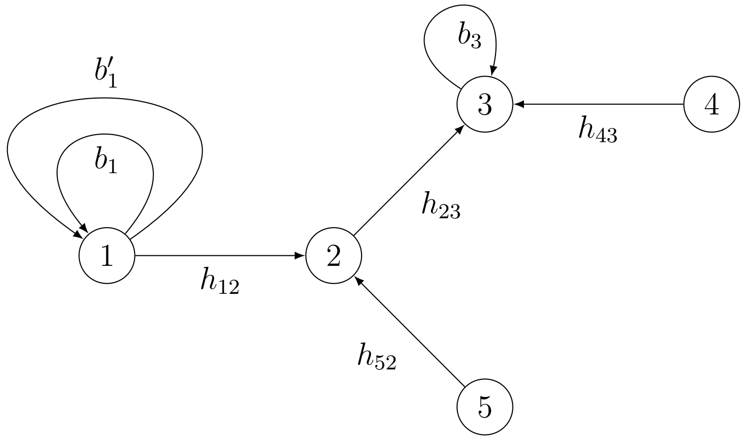


i.e. telles qu'il existe un drapeau de V stable par x et $\bar{x}$, x seulement agissant de manière nilpotente. Cette définition se généralise ainsi :

Définition 0.22. Une représentation x est dite *semi-nilpotente* s'il existe un drapeau I-gradué $\mathsf{W}_\bullet$ stable par x tel que $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}$ pour h boucle de Ω. On note $\Lambda(\alpha)$ la sous-variété des éléments semi-nilpotents de $\mu_\alpha^{-1}(0)$.

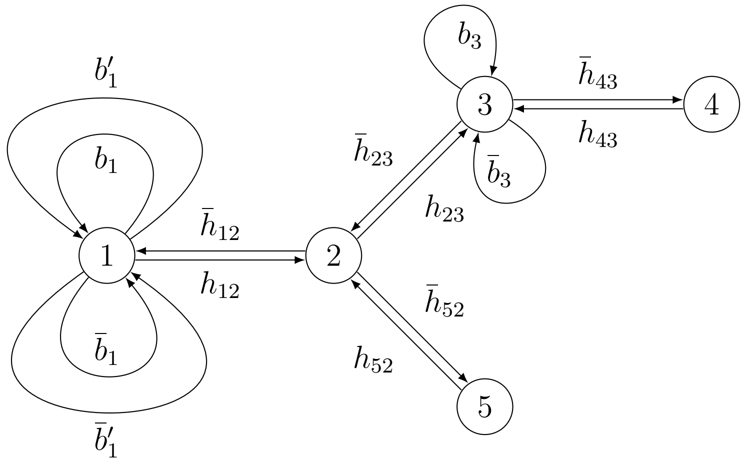
Si on considère par exemple :

$Q =$



alors :

$\bar{Q} =$



Ici, les représentations $x = (x_h)_{h\in H}$ sont semi-nilpotentes s'il existe un drapeau $\mathsf{W}_\bullet$ tel que :

- ▷ $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet$ si h est du type h_{ij} ou $\bar{h}_{ij}$, ce qui équivaut, puisqu'alors h n'est pas une boucle (voir 0.5), à $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}$;
- ▷ $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}$ si h est du type b_i ou b'_i ;
- ▷ $x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet$ si h est du type $\bar{b}_i$ ou $\bar{b}'_i$.

Le premier résultat obtenu est le suivant :

Théorème A. *Dans le cas du carquois à un sommet et plus de deux boucles, $\Lambda(\alpha) \subseteq \bar{E}_\alpha$ est Lagrangienne, et ses composantes irréductibles sont paramétrées par les compositions de α.*

On sait que dans le cas du carquois de Jordan, les composantes sont paramétrées par les partitions. On peut ensuite utiliser les résultats connus des carquois à un sommet pour obtenir le théorème suivant concernant les carquois généraux :

Théorème B. *Pour tous $i \in I$ et $\alpha \in \mathbb{N}I$, il existe une stratification :*

$$\Lambda(\alpha) = \sqcup_{l\geq 0}\Lambda(\alpha)_{i,l}$$

induisant un jeu de bijections :

$$\operatorname{Irr}\Lambda(\alpha)_{i,l} \xrightarrow{\sim} \operatorname{Irr}\Lambda(\alpha - li)_{i,0} \times \operatorname{Irr}\Lambda(li).$$

De plus, la sous-variété $\Lambda(\alpha) \subseteq \bar{E}_\alpha$ est Lagrangienne.

On peut alors obtenir un théorème strictement analogue à 0.8. Les bijections obtenues dans le théorème B montrent déjà que les opérateurs de Kashiwara usuels ne sont pas suffisants pour décrire la combinatoire des carquois à boucles puisque $\operatorname{Irr}\Lambda(li)$ peut-être non trivial. L'étude des variétés carquois de Nakajima peut alors donner une intuition plus précise de la définition des cristaux généralisés. On commence par définir des sous-variétés d'éléments semi-nilpotents $\mathfrak{L}(\mathsf{v},\mathsf{w}) \subseteq \mathfrak{M}(\mathsf{v},\mathsf{w})$, et on obtient :

Théorème C. *Pour tous $i \in I$ et $\mathsf{v} \in \mathbb{N}I$, il existe une stratification :*

$$\mathfrak{L}(\mathsf{v},\mathsf{w}) = \sqcup_{l\geq 0}\mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l}$$

induisant un jeu de bijections :

$$\operatorname{Irr}\mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l} \xrightarrow{\sim} \operatorname{Irr}\mathfrak{L}(\mathsf{v}-li,\mathsf{w})_{i,0} \times \operatorname{Irr}\Lambda(li)$$

sous les conditions usuelles s'il n'y a pas de boucle en i, sinon sous la condition :

$$\mathsf{w}_i + \sum_{(h:i\to j\neq i)\in H} \mathsf{v}_j > 0.$$

De plus, la sous-variété $\mathfrak{L}(\mathsf{v},\mathsf{w}) \subseteq \mathfrak{M}(\mathsf{v},\mathsf{w})$ est Lagrangienne.

On peut aussi définir un produit tensoriel géométrique de composantes irréductibles. On définit une sous-variété $\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}') \subset \mathfrak{M}(\mathsf{v}+\mathsf{v}',\mathsf{w}+\mathsf{w}')$ telle que :

Théorème D. *Il existe une bijection :*

$$\begin{aligned}\otimes : \operatorname{Irr}\mathfrak{L}(\mathsf{v},\mathsf{w}) \times \operatorname{Irr}\mathfrak{L}(\mathsf{v}',\mathsf{w}') &\xrightarrow{\sim} \operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}')\\ (X,X') &\mapsto X\otimes X',\end{aligned}$$

et si l'on note $\widetilde{\mathfrak{Z}}(\mathbf{v}) = \sqcup_{\mathsf{v}+\mathsf{v}'=\mathbf{v}}\widetilde{\mathfrak{Z}}(\mathsf{v},\mathsf{v}')$, il existe une stratification ($i \in I$) :

$$\widetilde{\mathfrak{Z}}(\mathbf{v}) = \bigsqcup_{l\geq 0}\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l}$$

qui induit les bijections suivantes ($i \in I$, $l > 0$) :

$$\operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l} \xrightarrow{\sim} \operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathbf{v}-l\alpha_i)_{i,0} \times \operatorname{Irr}\Lambda(li)$$

sous les mêmes conditions qu'au théorème C.

On peut dores et déjà définir ϵ_i via le diagramme suivant :

$$\begin{tikzcd}[column sep=small] \operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l} \arrow[rr] \arrow[dr, "\epsilon_i"'] & & \operatorname{Irr}\widetilde{\mathfrak{Z}}(\mathbf{v}-l\alpha_i)_{i,0} \times \operatorname{Irr}\Lambda(li) \arrow[dl, "\mathrm{pr}_2"] \\ & \operatorname{Irr}\Lambda(li) & \end{tikzcd}$$

où $\operatorname{Irr}\Lambda(li)$ est un singleton, l'ensemble des partitions de l ou l'ensemble des compostions de l suivant que i est sans boucles, avec une boucle ou avec plus de deux boucles. Si l'on définit de manière analogue ϵ_i sur $\operatorname{Irr}\mathfrak{L}(\mathsf{w})$ grâce au théorème C, on obtient le théorème suivant :

Théorème E. *Soit i un sommet présentant une ou plusieurs boucles, et* $(X, X') \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \operatorname{Irr} \mathfrak{L}(\mathsf{v}', \mathsf{w}')$. *On a :*

$$\epsilon_i(X \otimes X') = \begin{cases} \epsilon_i(X') & \text{si } \mathsf{w}'_i + \sum\limits_{(h:i\to j\neq i)\in H} \mathsf{v}'_{t(h)} > 0 \\ \epsilon_i(X) & \text{sinon.} \end{cases}$$

Tous ces résultats donnent assez de rigidité pour définir une version généralisée des cristaux de Kashiwara en fin de première partie, ainsi qu'un produit tensoriel.

0.5.2 Des faisceaux pervers et la conjecture de Lusztig

Dans une seconde partie, on commence par démontrer la conjecture 0.3 de Lusztig évoquée plus haut, en étendant les résultats connus dans le cas des carquois sans boucle et des carquois à un sommet et plusieurs boucles. On utilise, comme dans le cas classique, une récurrence sur α. Ici l'initialisation peut cependant être non triviale dans le cas $\alpha = li$ s'il y a une ou plusieurs boucles en i. Cependant ces cas sont connus, comme expliqué plus haut. La difficulté se situe en fait dans l'étude des foncteurs de restriction du type $\mathrm{Res}_{\alpha-li,li}$, quand i présente une ou plusieurs boucles. On introduit une notion d'invariance adaptée à cette fin, et une notion de régularité permettant une étude précise des supports des faisceaux considérés. On prouve ainsi le théorème suivant, analogue au théorème B :

Théorème F. *Pour tous $i \in I$ et $\alpha \in \mathbb{N}I$, il existe une stratification :*

$$\mathcal{P}_\alpha = \sqcup_{l\geq 0} \mathcal{P}_{\alpha,i,l}$$

induisant un jeu de bijections :

$$\mathcal{B}_{\alpha,i,l} \xrightarrow{\sim} \mathcal{B}_{\alpha-li,i,0} \times \mathcal{B}_{li}.$$

La conjecture de Lusztig est un corollaire de ce théorème.

Toutes les études géométriques faites permettent alors de définir une algèbre de Hopf généralisant les groupes quantiques usuellement associés aux carquois sans boucles.

On définit $\mathbf{U}_v^+$ par un générateur E_i à chaque sommet *réel* i (*i.e.* sans boucles), et une famille $(E_{i,l})_{l>0}$ à chaque sommet *imaginaire* i (*i.e.* avec boucle(s)), et on pose $\deg(E_{i,l}) = li$. En plus des relations de Serre usuelles, on impose des *relations de Serre de plus haut ordre* :

$$\sum_{t+t'=-la_{i,j}+1} (-1)^t E_j^{(t)} E_{i,l} E_j^{(t')} = 0$$

pour tout sommet réel j et tout sommet imaginaire i. D'après le cas du carquois de Jordan, on impose aussi :

$$[E_{i,l}, E_{i,k}] = 0$$

s'il n'y a qu'une boucle en i.

On peut alors prouver plusieurs propriétés intéressantes de cette algèbre, notamment un analogue du théorème de Gabber-Kac à propos de la non-dégénérescence des formes de Hopf sur $\mathbf{U}_v^+$ (via la définition notamment d'une quasi-$\mathcal{R}$-matrice et d'un opérateur de Casimir). On peut alors obtenir :

Théorème G. *On a un isomorphisme d'algèbres de Hopf :*

$$\mathbf{U}_v^+ \xrightarrow{\sim} \mathcal{K}.$$

En particulier, on remarque que $\mathcal{B}_{li}$ est l'ensemble des compositions de l si i présente plusieurs boucles. Ce résultat permet de généraliser la base canonique. À ce stade, il n'est par contre pas possible de généraliser la base semi-canonique, faute de résultats analogues à ceux obtenus dans [KS97], s'appuyant sur une étude fine de la théorie des cristaux, pas réalisée dans cette thèse. On a cependant une surjection :

$$\mathbf{U}^+(\mathfrak{g}) \twoheadrightarrow \mathcal{M}_\circ \underset{\text{libre}}{\supset} (f_Z)_{Z\in\operatorname{Irr}\Lambda}$$

et l'existence d'une famille libre $(f_Z)_{Z\in\operatorname{Irr}\Lambda}$ vérifiant les mêmes hypothèses que dans 0.8. On verra en conclusion comment résoudre ce problème, et de manière plus générale, comment étudier les nouveaux cristaux définis en fin de première partie, en s'appuyant notamment sur le groupe quantique généralisé défini dans la seconde.

Première partie

Quivers with loops and Lagrangian subvarieties

Sommaire

Introduction

Lusztig defined in [Lus91] Lagrangian subvarieties of the cotangent stack to the moduli stack of representations of a quiver associated to any Kac-Moody algebra. The proof of the Lagrangian character of these varieties was obtained via the study of some natural stratifications of each irreducible component, and then proceeding by induction. This particular structure on the set of irreducible components made it possible for Kashiwara and Saito in [KS97] to relate this variety to the usual quantum group associated to Kac-Moody algebras, via the notion of *crystals*. This later led Lusztig in [Lus00] to define a *semicanonical basis* of this quantum group, indexed by the irreducible components of these Lagrangian varieties.

There are more and more evidences of the relevance of the study of quivers with loops. A particular class of such quivers are the comet-shaped quivers, which have recently been used by Hausel, Letellier and Rodriguez Villegas in their study of the topology of character varieties, where the number of loops at the central vertex is the genus of the considered curve (see [HRV08] and [HLRV13]). We can also see quivers with loops appearing in a work of Nakajima relating quiver varieties with branching (see [Nak09]), as in the work of Okounkov and Maulik about quantum cohomology (see [MO12]).

Kang, Kashiwara and Schiffmann generalized these varieties in the framework of generalized Kac-Moody algebra in [KKS09], using quivers with loops. In this case, one has to impose a somewhat unnatural restriction on the regularity of the maps associated to the loops.

In this article we define a generalization of such Lagrangian varieties in the case of arbitrary quivers, possibly carrying loops. As opposed to the Lagrangian varieties constructed by Lusztig, which consisted in nilpotent representations, we have to consider here slightly more general representations. That this is necessary is already clear from the Jordan quiver case. Note that our Lagrangian variety is strictly larger than the one considered in [KKS09] and has many more irreducible components. Our proof of the Lagrangian character is also based on induction, but with non trivial first steps, consisting in the study of quivers with one vertex but possible loops. From our proof emerges a new combinatorial structure on the set of irreducible components, which is more general than the usual crystals, in that there are now more operators associated to a vertex with loops, see 1.14.

Then, we consider, following [Lus00], a convolution algebra of constructible functions on our varieties, and construct a family of constructible functions naturally attached to the irreducible components. In [Boz13b], we relate this convolution algebra to some explicit "Kac-Moody type" algebra, generalizing the notion of semicanonical basis.

In a second section, we construct Lagrangian subvarieties of Nakajima quiver varieties, still in the case of quivers with loops. In particular we get a geometric intuition of the way the tensor product of our generalized crystals should be defined (see section 3).

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1 Lusztig quiver varieties

Let Q be a quiver, defined by a set of vertices I and a set of oriented edges $\Omega = \{h : s(h) \to t(h)\}$. We denote by $\bar{h} : t(h) \to s(h)$ the opposite arrow of $h \in \Omega$, and $\bar{Q}$ the quiver $(I, H = \Omega \sqcup \bar{\Omega})$, where $\bar{\Omega} = \{\bar{h} \mid h \in \Omega\}$: each arrow is replaced by a pair of arrows, one in each direction, and we set $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$. We denote by $\Omega(i)$ the set of loops of Ω at i, and call i *imaginary* if $\omega_i = |\Omega(i)| \geq 1$, *real* otherwise. Denote by I^{im} (resp. I^{re}) the set of imaginary vertices (resp. real vertices). We work over the field of complex numbers $\mathbb{C}$.

For any pair of I-graded $\mathbb{C}$-vector spaces $V = (V_i)_{i\in I}$ and $V' = (V'_i)_{i\in I}$, we set :

$$\bar{E}(V, V') = \bigoplus_{h\in H} \mathrm{Hom}(V_{s(h)}, V'_{t(h)}).$$

For any dimension vector $\alpha = (\alpha_i)_{i\in I}$, we fix an I-graded $\mathbb{C}$-vector space V_α of dimension α, and put $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$. The space $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$ is endowed with a symplectic form :

$$\omega_\alpha(x, x') = \sum_{h\in H} \mathrm{Tr}(\epsilon(h) x_h x'_{\bar{h}})$$

which is preserved by the natural action of $G_\alpha = \prod_{i\in I} GL_{\alpha_i}(\mathbb{C})$ on $\bar{E}_\alpha$. The associated moment map $\mu_\alpha : \bar{E}_\alpha \to \mathfrak{g}_\alpha = \oplus_{i\in I} \mathrm{End}(V_\alpha)_i$ is given by :

$$\mu_\alpha(x) = \sum_{h\in H} \epsilon(h) x_{\bar{h}} x_h.$$

Here we have identified $\mathfrak{g}_\alpha^*$ with $\mathfrak{g}_\alpha$ via the trace pairing.

Definition 1.1. An element $x \in \bar{E}_\alpha$ is said to be *seminilpotent* if there exists an I-graded flag $\mathsf{W} = (\mathsf{W}_0 = \{0\} \subset \ldots \subset \mathsf{W}_r = V_\alpha)$ of V_α such that :

$$\begin{aligned} x_h(\mathsf{W}_\bullet) &\subseteq \mathsf{W}_{\bullet-1} \quad \text{if } h \in \Omega, \\ x_h(\mathsf{W}_\bullet) &\subseteq \mathsf{W}_\bullet \quad \text{if } h \in \bar{\Omega}. \end{aligned}$$

We put $\Lambda(\alpha) = \{x \in \mu_\alpha^{-1}(0) \mid x \text{ seminilpotent}\}$.

Lemma 1.2. *The variety $\Lambda(\alpha)$ is isotropic.*

Proof. We proceed as in [KKS09, 2.1], using the following general fact :

Proposition 1.3. *Let X be a smooth algebraic variety, Y a projective variety and Z a smooth closed algebraic subvariety of $X \times Y$. Consider the Lagrangian subvariety $\Lambda = T^*_Z(X \times Y)$ of $T^*(X \times Y)$. Then the image of the projection $q : \Lambda \cap (T^*X \times T^*_Y Y) \to T^*X$ is isotropic.*

We apply this result to $X = \oplus_{h\in\Omega} \mathrm{End}(V_{\alpha_{s(h)}}, V_{\alpha_{t(h)}})$, Y the I-graded flag variety of V_α and :

$$Z = \{(x, \mathsf{W}) \in X \times Y \mid x(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}\}.$$

In this case, we get :

$$\begin{aligned}
T^*X &= \bar{E}_\alpha \\
T^*Y &= \{(\mathsf{W},\xi) \in Y \times \mathfrak{g}_\alpha \mid \xi(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}\} \\
\Lambda &= \left\{ (x,\mathsf{W},\xi) \;\middle|\; \begin{array}{l} \xi = \sum_{h\in H} \epsilon(h) x_{\bar{h}} x_h \\ \forall h \in \Omega,\ x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1} \text{ and } x_{\bar{h}}(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\} \\
\operatorname{Im} q &= \left\{ x \in \bar{E}_\alpha \;\middle|\; \begin{array}{l} \mu_\alpha(x) = 0 \text{ and there exists } \mathsf{W} \in Y \text{ such that} \\ \forall h \in \Omega,\ x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1} \text{ and } x_{\bar{h}}(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\}
\end{aligned}$$

hence $\Lambda(\alpha) \subseteq \operatorname{Im} q$, which proves the lemma. □

1.1 The case of the Jordan quiver

This case is very well known. For $\alpha \in \mathbb{N}$, we have :

$$\Lambda(\alpha) = \{(x,y) \in (\operatorname{End}\mathbb{C}^\alpha)^2 \mid x \text{ nilpotent and } [x,y]=0\} = \bigcup_\lambda T^*_{\mathcal{O}_\lambda}(\operatorname{End}\mathbb{C}^\alpha),$$

where $\mathcal{O}_\lambda$ is the nilpotent orbit associated to the partition λ of α. Therefore $\Lambda(\alpha)$ is a Lagrangian subvariety of $(\operatorname{End}\mathbb{C}^\alpha)^2$, and its irreducible components are the closures of the conormal bundles to the nilpotent orbits.

1.2 The case of the quiver with one vertex and $g \geq 2$ loops

For $\alpha \in \mathbb{N}$, $\Lambda(\alpha)$ is the subvariety of $(\operatorname{End}\mathbb{C}^\alpha)^{2g}$ with elements $(x_i,y_i)_{1\leq i\leq g}$ such that :
▷ there exists a flag W of $\mathbb{C}^\alpha$ such that $x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}$ and $y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet$;
▷ $\sum_{1\leq i\leq g} [x_i,y_i] = 0$.

We will often forget the index $1 \leq i \leq g$ in the rest of this section, which is dedicated to the proof of the following theorem :

Theorem 1.4. *The subvariety $\Lambda(\alpha) \subseteq (\operatorname{End}\mathbb{C}^\alpha)^{2g}$ is Lagrangian. Its irreducible components are parametrized by the compositions $\mathsf{c} = (0 = \mathsf{c}_0 < \mathsf{c}_1 < \ldots < \mathsf{c}_r = \alpha)$ of α.*

Notations 1.5. For $(x_i,y_i) \in \Lambda(\alpha)$, we define $\mathsf{W}^0(x_i,y_i) = \mathbb{C}^\alpha$, then by induction $\mathsf{W}^{k+1}(x_i,y_i)$ the smallest subspace of $\mathbb{C}^\alpha$ containing $\sum x_i(\mathsf{W}^k(x_i,y_i))$ and stable by (x_i,y_i). By seminilpotency, we can define r to be the first power such that $\mathsf{W}^r(x_i,y_i) = \{0\}$. Although r depends on (x_i,y_i) we don't write it explicitly. We also set $\mathsf{W}_k(x_i,y_i) = \mathsf{W}^{r-k}(x_i,y_i)$.

Let :

$$\mathsf{c}(x_i,y_i) = (0 = \mathsf{c}_0(x_i,y_i) < \mathsf{c}_1(x_i,y_i) < \ldots < \mathsf{c}_r(x_i,y_i) = \alpha)$$

denotes the tuple of dimensions associated to the flag $\mathsf{W}_\bullet(x_i,y_i)$. For every composition $\mathsf{c} = (0 = \mathsf{c}_0 < \mathsf{c}_1 < \ldots < \mathsf{c}_r = \alpha)$ of α, we define a locally closed subvariety :

$$\Lambda(\mathsf{c}) = \{(x_i,y_i) \in \Lambda(\alpha) \mid \dim \mathsf{W}_\bullet(x_i,y_i) = \mathsf{c}\} \subseteq \Lambda(\alpha).$$

Then, if $\delta = (\delta_1, \ldots, \delta_{r-1}) \in \mathbb{N}^{r-1}$, let $\Lambda(\mathsf{c})_\delta \subseteq \Lambda(\mathsf{c})$ be the locally closed subvariety defined by :

$$\left(\dim\left(\bigcap_{1\leq i\leq g} \ker\left\{X \mapsto y_i^{(k+1)}X - Xy_i^{(k)}\right\}\right)\right)_{1\leq k\leq r-1} = \delta,$$

where :

$$y_i^{(k)} \in \mathrm{End}\left(\frac{\mathsf{W}_k(x_i,y_i)}{\mathsf{W}_{k-1}(x_i,y_i)}\right)$$

is induced by y_i and :

$$X \in \mathrm{Hom}\left(\frac{\mathsf{W}_k(x_i,y_i)}{\mathsf{W}_{k-1}(x_i,y_i)}, \frac{\mathsf{W}_{k+1}(x_i,y_i)}{\mathsf{W}_k(x_i,y_i)}\right).$$

Set $l = \mathsf{c}_r - \mathsf{c}_{r-1}$, then :

$$\check{\Lambda}(\mathsf{c})_\delta = \left\{(x_i,y_i,\mathfrak{X},\beta,\gamma) \;\middle|\; \begin{array}{l} (x_i,y_i) \in \Lambda(\mathsf{c})_\delta \\ \mathsf{W}_{r-1}(x_i,y_i) \oplus \mathfrak{X} = \mathbb{C}^\alpha \\ \beta : \mathsf{W}_{r-1}(x_i,y_i) \xrightarrow{\sim} \mathbb{C}^{\mathsf{c}_{r-1}} \text{ and } \gamma : \mathfrak{X} \xrightarrow{\sim} \mathbb{C}^l \end{array}\right\},$$

and :

$$\pi_{\mathsf{c},\delta} \left| \begin{array}{l} \check{\Lambda}(\mathsf{c})_\delta \to \Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g \\ (x_i,y_i,\mathfrak{X},\beta,\gamma) \mapsto (\beta_*(x_i,y_i)_{\mathsf{W}_{r-1}}, \gamma_*(y_i)_{\mathfrak{X}}) \end{array}\right.$$

where $\mathsf{c}^- = (\mathsf{c}_0 < \mathsf{c}_1 < \ldots < \mathsf{c}_{r-1})$ and $\delta^- = (\delta_1, \ldots, \delta_{r-2})$. Let finally $(\Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g)_{\mathsf{c},\delta}$ denotes the image of $\pi_{\mathsf{c},\delta}$.

Proposition 1.6. *The morphism $\pi_{\mathsf{c},\delta}$ is smooth over its image, with connected fibers of dimension $\alpha^2 + (2g-1)l(\alpha - l) + \delta_{r-1}$ whenever $\Lambda(\mathsf{c})_\delta \neq \varnothing$.*

Proof. Let $(x_i, y_i, z_i) \in (\Lambda(\mathsf{c}^-)_{\delta^-} \times (\mathrm{End}\,\mathbb{C}^l)^g)_{\mathsf{c},\delta}$. Let $\mathfrak{W}$ and $\mathfrak{X}$ be two supplementary subspaces of $\mathbb{C}^\alpha$ such that $\dim \mathfrak{X} = l$, together with two isomorphisms :

$$\beta : \mathfrak{W} \xrightarrow{\sim} \mathbb{C}^{\mathsf{c}_{r-1}} \text{ and } \gamma : \mathfrak{X} \xrightarrow{\sim} \mathbb{C}^l.$$

We identify x_i, y_i and z_i with $\beta^*(x_i, y_i)$ and $\gamma^* z_i$, and define an element (X_i, Y_i) in the fiber of (x_i, y_i, z_i) by setting :

$$\begin{aligned} (X_i,Y_i)_{\mathfrak{W}} &= (x_i,y_i) \\ (X_i,Y_i)_{\mathfrak{X}} &= (0,z_i) \\ (X_i,Y_i)_{|\mathfrak{X}}^{|\mathfrak{W}} &= (u_i,v_i) \in \mathrm{Hom}(\mathfrak{X},\mathfrak{W})^{2g}. \end{aligned}$$

Then :

$$\mu_\alpha(X_i,Y_i) = 0 \Leftrightarrow \phi(u_i,v_i) = \sum_{i=1}^g (x_iv_i + u_iz_i - y_iu_i) = 0,$$

and, for $X \in \mathrm{Hom}(\mathfrak{W}, \mathfrak{X})$:

$$\forall (u_i, v_i),\ \mathrm{Tr}(X\phi(u_i, v_i)) = 0 \Leftrightarrow \begin{cases} \forall i, \forall u_i,\ \mathrm{Tr}(X(u_i z_i - y_i u_i)) = 0 \\ \forall i, \forall v_i,\ \mathrm{Tr}(X x_i v_i) = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall i, \forall u_i,\ \mathrm{Tr}((z_i X - X y_i) u_i) = 0 \\ \forall i, \forall v_i,\ \mathrm{Tr}(X x_i v_i) = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall i,\ z_i X = X y_i \\ \forall i,\ X x_i = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} \mathsf{W}^1(x_i, y_i) = \mathsf{W}_{r-2}(x_i, y_i) \subseteq \ker X \\ \forall i,\ z_i X^{(r-1)} = X^{(r-1)} y_i^{(r-1)} \end{cases}$$

where $X^{(r-1)}$ denotes the map $\mathfrak{W}/\mathsf{W}_{r-2}(x_i, y_i) \to \mathfrak{X}$ induced by X. Since (x_i, y_i, z_i) is in the image of $\pi_{\mathsf{c},\delta}$, then the image of ϕ is of codimension δ_{r-1}, and thus its kernel is of dimension $(2g-1)l(\alpha - l) + \delta_{r-1}$.

Moreover, if we denote by $u_i^{(r-1)}$ the map $\mathfrak{X} \to \mathfrak{W}/\mathsf{W}_{r-2}(x_i, y_i)$ induced by u_i, $\mathsf{W}^1(X_i, Y_i) = \mathfrak{W}$ if and only if the space spanned by the action of $(y_i^{(r-1)})_i$ on $\sum_i \mathrm{Im}\, u_i^{(r-1)}$ is $\mathfrak{W}/\mathsf{W}_{r-2}(x_i, y_i)$. This condition defines an open subset of $\ker \phi$.

We end the proof noticing that the set of elements $(\mathfrak{W}, \mathfrak{X}, \beta, \gamma)$ is isomorphic to $GL_\alpha(\mathbb{C})$. □

Proposition 1.7. *The variety $\Lambda(\mathsf{c})_0$ is not empty.*

Proof. Fix W of dimension c and define x_1 such that

$$x_1(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet - 1}$$
$$x_1{}^{|\mathsf{W}_k/\mathsf{W}_{k-1}}_{|\mathsf{W}_{k+1}/\mathsf{W}_k} \neq 0.$$

We define inductively an element y_1 stabilizing W such that :

- ▷ the action of ${y_1}^{(k)}$ on $\mathrm{Im}\left(x_1{}^{|\mathsf{W}_k/\mathsf{W}_{k-1}}_{|\mathsf{W}_{k+1}/\mathsf{W}_k}\right)$ spans $\mathsf{W}_k/\mathsf{W}_{k-1}$;
- ▷ $\mathrm{Spec}\, {y_1}^{(k)} \cap \mathrm{Spec}\, {y_1}^{(k-1)} = \varnothing$.

We finally set $x_2 = -x_1, y_2 = y_1$ and $x_i = y_i = 0$ for $i > 2$. This yields an element (x_i, y_i) in $\Lambda(\mathsf{c})_0$. □

Corollary 1.8. *For any* c$= (0 = \mathsf{c}_0 < \mathsf{c}_1 < \ldots < \mathsf{c}_r = \alpha)$*, $\Lambda(\mathsf{c})_0$ is irreducible of dimension $g\alpha^2$.*

Proof. We argue by induction on r. If $\mathsf{c} = (0 = \mathsf{c}_0 < \mathsf{c}_1 = \alpha)$, we have $\Lambda(\mathsf{c})_0 = \Lambda(\mathsf{c}) = (\mathrm{End}\, \mathbb{C}^\alpha)^g$ which is irreducible of dimension $g\alpha^2$. For the induction step, 1.6 and 1.7 ensure us that $\check{\Lambda}(\mathsf{c})_0$ is irreducible of dimension :

$$\alpha^2 + (2g-1)l(\alpha - l) + \dim(\Lambda(\mathsf{c}^-)_0 \times (\mathrm{End}\, \mathbb{C}^l)^g)_{\mathsf{c},0}$$
$$= \alpha^2 + (2g-1)l(\alpha - l) + g(\alpha - l)^2 + gl^2$$

since $(\Lambda(\mathsf{c}^-)_0 \times (\operatorname{End}\mathbb{C}^l)^g)_{\mathsf{c},0}$ is a non empty subvariety of $\Lambda(\mathsf{c}^-)_0 \times (\operatorname{End}\mathbb{C}^l)^g$, irreducible of dimension $g(\alpha-l)^2 + gl^2$ by our induction hypothesis. Moreover,

$$\check{\Lambda}(\mathsf{c})_0 \to \Lambda(\mathsf{c})_0$$

being a principal bundle with fibers of dimension $\alpha^2 - l(\alpha - l)$, we get that $\Lambda(\mathsf{c})_0$ is irreducible of dimension

$$\alpha^2 + (2g-1)l(\alpha-l) + g(\alpha-l)^2 + gl^2 - \alpha^2 + l(\alpha-l) = g\alpha^2.$$

□

Lemma 1.9. *Let V and W be two vector spaces, and $k \geq 0$. For any $(u,v) \in \operatorname{End} V \times \operatorname{End} W$, we set :*

$$\begin{aligned} \mathcal{C}(u,v) &= \{x \in \operatorname{Hom}(V,W) \mid xu = vx\} \\ (\operatorname{End} V \times \operatorname{End} W)_k &= \{(u,v) \in \operatorname{End} V \times \operatorname{End} W \mid \dim \mathcal{C}(u,v) = k\}. \end{aligned}$$

Then we have

$$\operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k \geq k.$$

Proof. The restrictions of an endomorphism a to a generalized eigenspace associated to an eigenvalue η will be denoted by $a_\eta = \eta \operatorname{id} + \tilde{a}_\eta$. As usual, the nilpotent orbit associated to a partition ξ will be denoted by $\mathcal{O}_\xi$. We have :

$$\begin{aligned} \operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k &= \operatorname{codim}\{(u,v) \mid \textstyle\sum_{\alpha,\beta} \dim \mathcal{C}(u_\alpha, v_\beta) = k\} \\ &= \operatorname{codim}\{(u,v) \mid \textstyle\sum_{\alpha \in \operatorname{Spec} u \cap \operatorname{Spec} v} \dim \mathcal{C}(u_\alpha, v_\alpha) = k\} \\ &= \operatorname{codim}\{(u,v) \mid \textstyle\sum_\alpha \dim \mathcal{C}(\tilde{u}_\alpha, \tilde{v}_\alpha) = k\} \\ &= \operatorname{codim}\left\{(u,v) \,\middle|\, \begin{array}{l} (\tilde{u}_\alpha, \tilde{v}_\alpha) \in \mathcal{O}_{\lambda_\alpha} \times \mathcal{O}_{\mu_\alpha} \\ \sum_\alpha \sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j = k \end{array} \right\} \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{codim}(\operatorname{End} V \times \operatorname{End} W)_k \geq k \\ &\Leftrightarrow \sum_\alpha (\operatorname{codim} \mathcal{O}_{\lambda_\alpha} + \operatorname{codim} \mathcal{O}_{\mu_\alpha} - 1) \geq \sum_\alpha \textstyle\sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j \\ &\Leftrightarrow \sum_\alpha (\textstyle\sum_j (\lambda'_\alpha)_j^2 + \sum_j (\mu'_\alpha)_j^2 - 1) \geq \sum_\alpha \textstyle\sum_j (\lambda'_\alpha)_j (\mu'_\alpha)_j \end{aligned}$$

which is clear. □

Proposition 1.10. *If $\delta \neq 0$, we have* $\dim \Lambda(\mathsf{c})_\delta < g\alpha^2$.

Proof. It's enough to show that if $\delta_{r-1} > 0$, we have :

$$\dim(\Lambda(\mathsf{c}^-)_{\delta^-} \times (\operatorname{End}\mathbb{C}^l)^g)_{\mathsf{c},\delta} + \delta_{r-1} < \dim(\Lambda(\mathsf{c}^-)_0 \times (\operatorname{End}\mathbb{C}^l)^g).$$

This is a consequence of the previous lemma (recall that $g \geq 2$). Indeed, if we set :

$$((\operatorname{End} V)^g \times (\operatorname{End} W)^g)_k = \{(u_i, v_i) \mid \dim \cap_i \mathcal{C}(u_i, v_i) = k\},$$

we have :

$$((\mathrm{End}\,V)^g \times (\mathrm{End}\,W)^g)_k \subseteq \prod_{i=1}^{g} (\mathrm{End}\,V \times \mathrm{End}\,W)_{k_i}$$

for some $k_i \geq k$, and thus :

$$\mathrm{codim}((\mathrm{End}\,V)^g \times (\mathrm{End}\,W)^g)_k \geq \sum_i \mathrm{codim}(\mathrm{End}\,V \times \mathrm{End}\,W)_{k_i} \geq \sum_i k_i \geq gk > k.$$

□

The following proposition concludes the proof of theorem 1.4 :

Proposition 1.11. *Every irreducible component of* $\Lambda(\mathsf{c})$ *is of dimension larger than or equal to* $g\alpha^2$.

Proof. We first prove the result for the following variety :

$$\tilde{\Lambda}(\mathsf{c}) = \{((x_i, y_i), \mathsf{W}) \in \Lambda(\alpha) \times Y_{\mathsf{c}} \mid x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1} \text{ and } y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet\}$$

where Y_{c} denotes the variety of flags of $\mathbb{C}^\alpha$ of dimension w. We use the following notations, analogous to 1.2 :

$$\begin{aligned} X &= \{(x_i)_{1\leq i\leq g} \in (\mathrm{End}\,\mathbb{C}^\alpha)^g\} \\ Z &= \{((x_i)_{1\leq i\leq g}, \mathsf{W}) \mid x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}\} \subseteq X \times Y_{\mathsf{c}}. \end{aligned}$$

We get :

$$\begin{aligned} T^*X &= \{(x_i, y_i)_{1\leq i\leq g} \in (\mathrm{End}\,\mathbb{C}^\alpha)^{2g}\} \\ T^*Y_{\mathsf{c}} &= \{(\mathsf{W}, K) \in Y_{\mathsf{c}} \times \mathrm{End}\,\mathbb{C}^\alpha \mid K(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1}\} \end{aligned}$$

and :

$$T^*_Z(X \times Y_{\mathsf{c}}) = \left\{ ((x_i, y_i), \mathsf{F}, K) \;\middle|\; \begin{array}{l} \sum_{1\leq i\leq g} [x_i, y_i] = K \\ x_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1} \text{ and } y_i(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet \end{array} \right\}$$

which is a pure Lagrangian subvariety of $T^*(X \times Y_{\mathsf{c}})$, of dimension $g\alpha^2 + \dim Y_{\mathsf{c}}$. Since T^*Y_{c} is irreducible of dimension $2\dim Y_{\mathsf{c}}$, the irreducible components of the fibers of $T^*_Z(X \times Y_{\mathsf{c}}) \to T^*Y_{\mathsf{c}}$ are of dimension larger than or equal to $g\alpha^2 - \dim Y_{\mathsf{c}}$. We denote by $\tilde{\Lambda}_{\mathsf{W}}$ the fiber above $(\mathsf{W}, 0)$, and by P the stabilizer of W in G_α. Since G_α and P are irreducible, we get that the components of :

$$\tilde{\Lambda}(\mathsf{c}) = G_\alpha \times_P \tilde{\Lambda}_{\mathsf{W}}$$

are of dimension larger than or equal to $\dim Y_{\mathsf{c}} + (g\alpha^2 - \dim Y_{\mathsf{c}}) = g\alpha^2$.

We extend this result to $\Lambda(\mathsf{c})$, noticing that :

$$\begin{array}{rcl} \Lambda(\mathsf{c}) & \hookrightarrow & \tilde{\Lambda}(\mathsf{c}) \\ (x_i, y_i) & \mapsto & (x_i, y_i, \mathsf{W}_\bullet(x_i, y_i)) \end{array}$$

defines an open embedding. □

1.3 The general case

For every $\alpha, \beta \in \mathbb{N}^I$ and $j \in I$, we put :

$$\begin{aligned}(\alpha, \beta) &= \sum_{h \in \Omega} \alpha_{s(h)} \beta_{t(h)} \\ \langle \alpha, \beta \rangle &= \sum_{i \in I} \alpha_i \beta_i \\ e_j &= (\delta_{i,j})_{i \in I}.\end{aligned}$$

Definition 1.12. For every subset $A \subseteq I$, and every $x \in \Lambda(\alpha)$, we denote by $\mathfrak{I}_A(x)$ the subspace of V_α spanned by the action of x on $\oplus_{i \notin A} V_i$. Then, for $\underline{l} = (l_i)_{i \in A}$, we set :

$$\Lambda(\alpha)_{A,\underline{l}} = \{x \in \Lambda(\alpha) \mid \operatorname{codim} \mathfrak{I}_A(x) = \underline{l}\}.$$

In the case where A is a singleton $\{i\}$, $\underline{l}$ is of the form $(\delta_{i,j} l)_{j \in I}$ and we write $\Lambda(\alpha)_{i,l}$ instead of $\Lambda(\alpha)_{\{i\},\underline{l}}$.

Remark 1.13. By the definition of seminilpotency, we have :

$$\Lambda(\alpha) = \bigcup_{i \in I, l \geq 1} \Lambda(\alpha)_{i,l}.$$

Indeed, if $x \in \Lambda(\alpha)$, there exists an I-graded flag $(\mathsf{W}_0 \subsetneq \ldots \subsetneq \mathsf{W}_r = \mathbb{C}^n)$ such that (x, W) satisfies 1.1. Therefore there exists $i \in I$ and $l > 0$ such that $\mathsf{W}_r/\mathsf{W}_{r-1} \simeq V_{le_i}$, and thus $x \in \Lambda(\alpha)_{i,l}$.

Proposition 1.14. *There exists a variety* $\check{\Lambda}(\alpha)_{A,\underline{l}}$ *and a diagram :*

$$\begin{tikzcd} & \check{\Lambda}(\alpha)_{A,\underline{l}} \arrow[dl, "q_{A,\underline{l}}"'] \arrow[dr, "p_{A,\underline{l}}"] & \\ \Lambda(\alpha)_{A,\underline{l}} & & \Lambda(\alpha - \underline{l})_{A,\underline{0}} \times \Lambda(\underline{l}) \end{tikzcd}$$

such that $p_{A,\underline{l}}$ *and* $q_{A,\underline{l}}$ *are smooth with connected fibers, inducing a bijection :*

$$\operatorname{Irr} \Lambda(\alpha)_{A,\underline{l}} \xrightarrow{\sim} \operatorname{Irr} \Lambda(\alpha - \underline{l})_{A,\underline{0}} \times \operatorname{Irr} \Lambda(\underline{l}).$$

Proof. In this proof we will denote by $I(V, V')$ the set of I-graded isomorphisms between two I-graded spaces V and V' of same I-graded dimension. We set :

$$\check{\Lambda}(\alpha)_{A,\underline{l}} = \left\{ (x, \mathfrak{X}, \beta, \gamma) \;\middle|\; \begin{array}{l} x \in \Lambda(\alpha)_{A,\underline{l}} \\ \mathfrak{X} \ I\text{-graded and } \mathfrak{I}_A(x) \oplus \mathfrak{X} = V_\alpha \\ \beta \in I(\mathfrak{I}_A(x), V_{\alpha - \underline{l}}) \text{ and } \gamma \in I(\mathfrak{X}, V_{\underline{l}}) \end{array} \right\}$$

and :

$$p_{A,\underline{l}} \left| \begin{array}{l} \check{\Lambda}(\alpha)_{A,\underline{l}} \to \Lambda(\alpha - \underline{l})_{A,\underline{0}} \times \Lambda(\underline{l}) \\ (x, \mathfrak{X}, \beta, \gamma) \mapsto (\beta_*(x_{\mathfrak{I}_A(x)}), \gamma_*(x_{\mathfrak{X}})). \end{array} \right.$$

We study the fibers of $p_{A,\underline{l}}$: take $y \in \Lambda(\alpha - \underline{l})_{A,\underline{0}}$ and $z \in \Lambda(\underline{l})$ and consider $\mathfrak{I}$ and $\mathfrak{X}$ two supplementary I-graded subspaces of V_α such that $\dim \mathfrak{X} = \underline{l}$, together with two isomorphisms :

$$\beta \in I(\mathfrak{I}, V_{\alpha-\underline{l}}) \text{ and } \gamma \in I(\mathfrak{X}, V_{\underline{l}}).$$

We identify y and z with $\beta^* y$ and $\gamma^* z$, and we define a preimage x by setting $x_{|\mathfrak{I}}^{|\mathfrak{I}} = y$, $x_{|\mathfrak{X}}^{|\mathfrak{X}} = z$ and $x_{|\mathfrak{X}}^{|\mathfrak{I}} = \eta \in \bar{E}(\mathfrak{X}, \mathfrak{I})$. In order to get $\mu_\alpha(x) = 0$, η must satisfy the following relation for every $i \in I$:

$$\phi_i(\eta) = \sum_{h \in H : s(h) = i} \epsilon(h)(y_{\bar{h}}\eta_h + \eta_{\bar{h}} z_h) = 0.$$

We need to show that $\phi = \oplus_{i \in I} \phi_i$ is surjective to conclude. Consider $\xi \in \oplus_{i \in I} \operatorname{Hom}(\mathfrak{I}_i, \mathfrak{X}_i)$ such that for every η :

$$\sum_{i \in I} \operatorname{Tr}(\phi_i(\eta)\xi_i) = 0.$$

Then we have for every edge h such that $s(h) = i$, $t(h) = j$ and for every η_h :

$$\operatorname{Tr}(y_{\bar{h}}\eta_h\xi_i) - \operatorname{Tr}(\eta_h z_{\bar{h}}\xi_j) = 0.$$

But the member of the left is equal to :

$$\operatorname{Tr}(\eta_h\xi_i y_{\bar{h}}) - \operatorname{Tr}(\eta_h z_{\bar{h}}\xi_j) = \operatorname{Tr}(\eta_h(\xi_i y_{\bar{h}} - z_{\bar{h}}\xi_j)),$$

hence $\xi_i y_{\bar{h}} = z_{\bar{h}}\xi_j$ and therefore $\ker \xi$ is stable by y. Moreover, $\mathfrak{X}_i = \{0\}$ if $i \notin A$ so $\ker \xi_i = \mathfrak{I}_i$ if $i \notin A$. As $\operatorname{codim} \mathfrak{I}_A(y) = \underline{0}$, we get $\xi = 0$, which finishes the proof. □

We can now state the following theorem, which answers a question asked in [Li] :

Theorem 1.15. *The subvariety $\Lambda(\alpha)$ of $\bar{E}_\alpha$ is Lagrangian.*

Proof. Since this subvariety is isotropic by 1.2 we just have to show that the irreducible components of $\Lambda(\alpha)$ are of dimension (α, α). We proceed by induction on α, the first step corresponding to the one vertex quiver case which has already been treated : we have seen that $\Lambda(le_i)$ is of dimension (le_i, le_i).

Next, consider $C \in \operatorname{Irr} \Lambda(\alpha)$ for some α. By 1.13, there exists $i \in I$ and $l \geq 1$ such that $C \cap \Lambda(\alpha)_{i,l}$ is dense in C. Let $\check{C} = (C_1, C_2)$ the couple of irreducible components corresponding to C via the bijection obtained in 1.14 in the case $A = \{i\}$ and $\underline{l} = le_i$:

$$\operatorname{Irr} \Lambda(\alpha)_{i,l} \xrightarrow{\sim} \operatorname{Irr} \Lambda(\alpha - le_i)_{i,0} \times \operatorname{Irr} \Lambda(le_i).$$

We also know by the proof of 1.14 that the fibers of $p_{A,\underline{l}}$ are of dimension :

$$\langle \alpha, \alpha \rangle + (\alpha - \underline{l}, \underline{l}) + (\underline{l}, \alpha - \underline{l}) - \langle \underline{l}, \alpha - \underline{l} \rangle.$$

Since $q_{A,\underline{l}}$ is a principal bundle with fibers of dimension $\langle \alpha, \alpha \rangle - \langle \underline{l}, \alpha - \underline{l} \rangle$, we get :

$$\dim C = \dim \check{C} + (\alpha - \underline{l}, \underline{l}) + (\underline{l}, \alpha - \underline{l}) = \dim \check{C} + (\alpha - le_i, le_i) + (le_i, \alpha - le_i).$$

But $\Lambda(\alpha - le_i)_{i,0}$ is open in $\Lambda(\alpha - le_i)$, so we can use our induction hypothesis and the first step to write :

$$\dim \check{C} = (\alpha - le_i, \alpha - le_i) + l^2(e_i, e_i)$$

and thus obtain :

$$\dim C = (\alpha - le_i, \alpha - le_i) + l^2(e_i, e_i) + (\alpha - le_i, le_i) + (le_i, \alpha - le_i) = (\alpha, \alpha).$$

□

1.4 Constructible functions

Following [Lus00], we denote by $\mathcal{M}(\alpha)$ the $\mathbb{Q}$-vector space of constructible functions $\Lambda(\alpha) \to \mathbb{Q}$, which are constant on any G_α-orbit. Put $\mathcal{M} = \oplus_{\alpha \geq 0}\mathcal{M}(\alpha)$, which is a graded algebra once equipped with the product $*$ defined in [Lus00, 2.1].

For $Z \in \operatorname{Irr}\Lambda(\alpha)$ and $f \in \mathcal{M}(\alpha)$, we put $\rho_Z(f) = c$ if $Z \cap f^{-1}(c)$ is an open dense subset of Z.

If $i \in I^{\text{im}}$ and (l) denotes the trivial composition or partition of l, we denote by $1_{i,l}$ the characteristic function of the associated irreducible component $Z_{i,(l)} \in \operatorname{Irr}\Lambda(le_i)$ (the component of elements x such that $x_h = 0$ for any loop $h \in \Omega$). If $i \notin I^{\text{im}}$, we just denote by 1_i the function mapping to 1 the only point in $\Lambda(e_i)$.

We have $1_{i,l} \in \mathcal{M}(le_i)$ for $i \in I^{\text{im}}$ and $1_i \in \mathcal{M}(e_i)$ for $i \notin I^{\text{im}}$. We denote by $\mathcal{M}_\circ \subseteq \mathcal{M}$ the subalgebra generated by these functions.

Lemma 1.16. *Suppose Q has one vertex $\circ$ and $g \geq 1$ loop(s). For every $Z \in \operatorname{Irr}\Lambda(\alpha)$ there exists $f \in \mathcal{M}_\circ(\alpha)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ for $Z' \neq Z$.*

Proof. We denote by Z_{c} the irreducible component associated to the partition (resp. composition) c of α ig $g = 1$ (resp. $g \geq 2$). By convention, if $g = 1$, Z_{c} will denote the component associated to the orbit $\mathcal{O}_{\mathsf{c}}$ défined by :

$$x \in \mathcal{O}_{\mathsf{c}} \Leftrightarrow \dim\ker x^i = \sum_{1 \leq k \leq i} \mathsf{c}_k,$$

where we see now compositions as (non ordered) tuples of $\mathbb{N}_{>0}$. If $g \geq 2$, we remark that by trace duality, we can assume that Z_{c} is the closure of $\breve{\Lambda}_{\mathsf{c}}$ defined by :

$$(x_i, y_i)_{1 \leq i \leq g} \in \breve{\Lambda}_{\mathsf{c}} \Leftrightarrow \dim \mathrm{K}_i = \sum_{1 \leq k \leq i} \mathsf{c}_k$$

where we define by induction $\mathrm{K}_0 = \{0\}$, then K_{j+1} as the biggest subspace of $\cap_i x_i^{-1}(\mathrm{K}_j)$ stable by (x_i, y_i). From now on, $\mathsf{c} = (\mathsf{c}_1, \ldots, \mathsf{c}_r)$ will denote indistinctly a partition or a composition depending on the value of g. We define an order by :

$$\mathsf{c} \preceq \mathsf{c}' \text{ if and only if for any } i \geq 1 \text{ we have } \sum_{1 \leq k \leq i} \mathsf{c}_k \leq \sum_{1 \leq k \leq i} \mathsf{c}'_k.$$

Therefore, setting $\tilde{1}_{\mathsf{c}} = 1_{\mathsf{c}_r} * \cdots * 1_{\mathsf{c}_1}$, where $1_l = 1_{\circ,l}$, we get :

$$x \in Z_{\mathsf{c}},\ \tilde{1}_{\mathsf{c}'}(x) \neq 0 \ \Rightarrow\ \mathsf{c}' \preceq \mathsf{c}.$$

For $\mathsf{c} = (\alpha)$ we have $\tilde{1}_{\mathsf{c}} = 1_\alpha$ which is the characteristic function of Z_{c}, and we put $1_{\mathsf{c}} = \tilde{1}_{\mathsf{c}}$ in this case. Then, by induction :

$$1_{\mathsf{c}} = \tilde{1}_{\mathsf{c}} - \sum_{\mathsf{c}' \prec \mathsf{c}} \rho_{Z_{\mathsf{c}'}}(\tilde{1}_{\mathsf{c}}) 1_{\mathsf{c}'}$$

has the expected property. $\square$

Notations 1.17. ▷ From now on, if c corresponds to an irreducible component of $\Lambda(|\mathsf{c}|e_i)$, we will note $1_{i,\mathsf{c}}$ the function corresponding to 1_{c} in the previous proof.

▷ For $Z \in \operatorname{Irr}\Lambda(\alpha)_{i,l}$, we denote by $\epsilon_i(Z) \in \operatorname{Irr}\Lambda(le_i)$ the composition of the second projection with the bijection obtained in 1.14 in the case $(A, \underline{l}) = (i, l)$. We also set $|\epsilon_i(Z)| = l$.

Proposition 1.18. *For every $Z \in \operatorname{Irr}\Lambda(\alpha)$, there exists $f \in \mathcal{M}_\circ(\alpha)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ if $Z' \neq Z$.*

Proof. We proceed as in [Lus00, lemma 2.4], by induction α. The first step consists in 1.16. Then, consider $Z \in \operatorname{Irr}\Lambda(\alpha)$. There exists $i \in I$ and $l > 0$ such that $Z \cap \Lambda(\alpha)_{i,l}$ is dense in Z.

We know proceed by descending induction on l. There's nothing to say if $l > \alpha_i$.

Otherwise, let $(Z', Z_{\mathsf{c}}) \in \operatorname{Irr}\Lambda(\alpha - le_i)_{i,0} \times \operatorname{Irr}\Lambda(le_i)$ be the pair of components corresponding to Z, then, by induction hyopthesis on α, there exists $g \in \mathcal{M}_\circ(\alpha - le_i)$ such that $\rho_{\overline{Z'}}(g) = 1$ and $\rho_Y(g) = 0$ if $\overline{Z}' \neq Y \in \operatorname{Irr}\Lambda(\alpha - le_i)$.

Then we set $\tilde{f} = 1_{i,\mathsf{c}} * g \in \mathcal{M}_\circ(\alpha)$, and get :
- $\rho_Z(\tilde{f}) = 1$,
- $\rho_{Z'}(\tilde{f}) = 0$ if $Z' \in \operatorname{Irr}\Lambda(\alpha) \setminus Z$ satisfies $|\epsilon_i(Z')| = l$,
- $\tilde{f}(x) = 0$ if $x \in \Lambda(\alpha)_{i,<l}$ so that $\rho_{Z'}(\tilde{f}) = 0$ if $|\epsilon_i(Z')| < l$.

If $|\epsilon_i(Z')| > l$, we use the induction hypothesis on l : there exists $f_{Z'} \in \mathcal{M}_\circ(\alpha)$ such that $\rho_{Z'}(f_{Z'}) = 1$ and $\rho_{Z''}(f_{Z'}) = 0$ if $Z'' \in \operatorname{Irr}\Lambda(\alpha) \backslash Z'$. We end the proof by setting :

$$f = \tilde{f} - \sum_{Z' : |\epsilon_i(Z')| > l} \rho_{Z'}(\tilde{f}) f_{Z'}.$$

□

2 Nakajima quiver varieties

Fix an I-graded vector space W of dimension $\mathsf{w} = (\mathsf{w}_i)_{i\in I}$. For any dimension vector $\mathsf{v} = (\mathsf{v}_i)_{i\in I}$, we still fix an I-graded $\mathbb{C}$-vector space $V_{\mathsf{v}} = ((V_{\mathsf{v}})_i = V_{\mathsf{v}_i})_{i\in I}$ of dimension v. We will denote by $(x, f, g) = ((x_h)_{h\in H}, (f_i)_{i\in I}, (g_i)_{i\in I})$ the elements of the following space :

$$E(V, \mathsf{w}) = \bar{E}(V, V) \oplus \bigoplus_{i\in I} \operatorname{Hom}(V_i, W_i) \bigoplus_{i\in I} \operatorname{Hom}(W_i, V_i)$$

defined for any I-graded space V, and put $E_{\mathsf{v},\mathsf{w}} = E(V_{\mathsf{v}}, \mathsf{w})$ for any dimension vector v. This space is endowed with a symplectic form :

$$\omega_{\mathsf{v},\mathsf{w}}\big((x, f, g), (x', f', g')\big) = \sum_{h\in H} \operatorname{Tr}(\epsilon(h) x_h x'_{\bar{h}}) + \sum_{i\in I} \operatorname{Tr}(g_i f'_i - g'_i f_i)$$

which is preserved by the natural action of $G_{\mathsf{v}} = \prod_{i\in I} GL_{\mathsf{v}_i}(\mathbb{C})$ on $E_{\mathsf{v},\mathsf{w}}$. The associated moment map $\mu_{\mathsf{v},\mathsf{w}} : E_{\mathsf{v},\mathsf{w}} \to \mathfrak{g}_{\mathsf{v}} = \oplus_{i\in I} \operatorname{End}(V_{\mathsf{v}})_i$ is given by :

$$\mu_{\mathsf{v},\mathsf{w}}(x, f, g) = \Big(g_i f_i + \sum_{h\in H : s(h) = i} \epsilon(h) x_{\bar{h}} x_h\Big)_{i\in I}.$$

Here we have identified $\mathfrak{g}_{\mathsf{v}}^*$ with $\mathfrak{g}_{\mathsf{v}}$ via the trace pairing. Put :

$$\mathrm{M}_\circ(\mathsf{v}, \mathsf{w}) = \mu_{\mathsf{v},\mathsf{w}}^{-1}(0).$$

Definition 2.1. Set $\chi : G_{\mathsf{v}} \to \mathbb{C}^*$, $(g_i)_{i\in I} \mapsto \prod_{i\in I} \det^{-1} g_i$. We denote by :

$$\mathfrak{M}_\circ(\mathsf{v},\mathsf{w}) = \mu_{\mathsf{v},\mathsf{w}}^{-1}(0)/\!\!/G_{\mathsf{v}}$$
$$\mathfrak{M}(\mathsf{v},\mathsf{w}) = \mu_{\mathsf{v},\mathsf{w}}^{-1}(0)/_\chi G_{\mathsf{v}}$$

the geometric and symplectic quotients (with respect to χ).

Proposition 2.2. *An element* $(x,f,g) \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w})$ *is stable with respect to* χ *if and only if the only* x*-stable subspace of* $\ker f$ *is* $\{0\}$. *Set :*

$$\mathrm{M}(\mathsf{v},\mathsf{w}) = \{(x,f,g) \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w}) \mid (x,f,g) \text{ stable}\}.$$

Definition 2.3. An element $(x,f,g) \in E_{\mathsf{v},\mathsf{w}}$ is said to be *seminilpotent* if $x \in \bar{E}_{\mathsf{v}}$ is, according to 1.1. We put :

$$\mathrm{L}_\circ(\mathsf{v},\mathsf{w}) = \{(x,f,0) \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w}) \mid x \text{ seminilpotent}\} \subseteq \mathrm{M}_\circ(\mathsf{v},\mathsf{w})$$

and define $\mathrm{L}(\mathsf{v},\mathsf{w}) \subseteq \mathrm{M}(\mathsf{v},\mathsf{w})$ in the same way. Finally set :

$$\mathfrak{L}_\circ(\mathsf{v},\mathsf{w}) = \mathrm{L}_\circ(\mathsf{v},\mathsf{w})/\!\!/G_{\mathsf{v}}$$
$$\mathfrak{L}(\mathsf{v},\mathsf{w}) = \mathrm{L}_\circ(\mathsf{v},\mathsf{w})/_\chi G_{\mathsf{v}} = \mathrm{L}(\mathsf{v},\mathsf{w})/\!\!/G_{\mathsf{v}}.$$

We will simply denote by (x,f) the elements of $\mathrm{L}_\circ(\mathsf{v},\mathsf{w})$.

Lemma 2.4. *The variety* $\mathrm{L}(\mathsf{v},\mathsf{w})$ *is isotropic.*

Proof. Analogous to 1.2. □

2.1 A crystal-type structure

For every $\mathsf{v},\mathsf{v}' \in \mathbb{N}^I$ and $j \in I$, we put :

$$(\mathsf{v},\mathsf{v}') = \sum_{h\in\Omega} \mathsf{v}_{s(h)}\mathsf{v}'_{t(h)}$$
$$\langle \mathsf{v},\mathsf{v}'\rangle = \sum_{i\in I} \mathsf{v}_i\mathsf{v}'_i$$
$$e_j = (\delta_{i,j})_{i\in I}.$$

Definition 2.5. For every subset $i \in I$, and every $(x,f,g) \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w})$, we denote by $\mathfrak{I}_i(x,f,g)$ the subspace of V_{v_i} spanned by the action of $x \oplus g$ on $(\oplus_{j\neq i} V_j) \oplus W_i$. Then, for $l \geq 0$, we set :

$$\mathrm{M}_\circ(\mathsf{v},\mathsf{w})_{i,l} = \{x \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w}) \mid \operatorname{codim} \mathfrak{I}_i(x,f,g) = l\}.$$

We define $\mathrm{M}(\mathsf{v},\mathsf{w})_{i,l}$, $\mathrm{L}_\circ(\mathsf{v},\mathsf{w})_{i,l}$ and $\mathrm{L}(\mathsf{v},\mathsf{w})_{i,l}$ in the same way. The quantity $\operatorname{codim} \mathfrak{I}_i(x,f,g)$ being stable on G_{v}-orbits, the notations $\mathfrak{M}_\circ(\mathsf{v},\mathsf{w})_{i,l}$, $\mathfrak{M}(\mathsf{v},\mathsf{w})_{i,l}$, $\mathfrak{L}_\circ(\mathsf{v},\mathsf{w})_{i,l}$ and $\mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l}$ make also sense.

Remark 2.6.

– By the definition of seminilpotency, we have :

$$\mathrm{L}_\circ(\mathsf{v},\mathsf{w}) = \bigsqcup_{i\in I, l\geq 1} \mathrm{L}_\circ(\mathsf{v},\mathsf{w})_{i,l}.$$

Indeed, if $(x,f) \in \mathrm{L}_\circ(\mathsf{v},\mathsf{w})$, there exists an I-graded flag $(F_0 \subsetneq \ldots \subsetneq F_r = V)$ such that (x,F) satisfies 1.1. Therefore there exists $i \in I$ and $l > 0$ such that $F_r/F_{r-1} \simeq V_i$, and thus $(x,f) \in \mathrm{L}_\circ(\mathsf{v},\mathsf{w})_{i,l}$.

– Note that $\mathrm{L}_\circ(le_i, 0) = \Lambda(le_i)$.

Proposition 2.7. *There exists a variety* $\check{\mathrm{M}}_\circ(\mathsf{v},\mathsf{w})_{i,l}$ *and a diagram :*

$$\begin{tikzcd} & \check{\mathrm{M}}_\circ(\mathsf{v},\mathsf{w})_{i,l} \arrow[dl, "q_{i,l}"'] \arrow[dr, "p_{i,l}"] & \\ \mathrm{M}_\circ(\mathsf{v},\mathsf{w})_{i,l} & & \mathrm{M}_\circ(\mathsf{v}-le_i,\mathsf{w})_{i,0} \times \mathrm{M}_\circ(le_i,0) \end{tikzcd} \tag{2.8}$$

such that $p_{i,l}$ *and* $q_{i,l}$ *are smooth with connected fibers, inducing a bijection :*

$$\mathrm{Irr}\,\mathrm{M}_\circ(\mathsf{v},\mathsf{w})_{i,l} \xrightarrow{\sim} \mathrm{Irr}\,\mathrm{M}_\circ(\mathsf{v}-le_i,\mathsf{w})_{i,0} \times \mathrm{Irr}\,\mathrm{M}_\circ(le_i,0).$$

The dimension of the fibers of $p_{i,l}$ *is :*

$$(le_i, \mathsf{v}-le_i) + (\mathsf{v}-le_i, le_i) + \langle \mathsf{v},\mathsf{v}\rangle - \langle le_i, \mathsf{v}-\mathsf{w}-le_i\rangle.$$

Proof. In this proof we will denote by $I(V,V')$ the set of I-graded isomorphisms between two I-graded spaces V and V' of same I-graded dimension. We set :

$$\check{\mathrm{M}}_\circ(\mathsf{v},\mathsf{w})_{i,l} = \left\{ (x,f,g,\mathfrak{X},\beta,\gamma) \;\middle|\; \begin{array}{l} (x,f,g) \in \mathrm{M}_\circ(\mathsf{v},\mathsf{w})_{i,l} \\ \mathfrak{X}\ I\text{-graded and } \mathfrak{I}_i(x,f,g) \oplus \mathfrak{X} = V_{\mathsf{v}} \\ \beta \in I(\mathfrak{I}_i(x,f,g), V_{\mathsf{v}-le_i}) \\ \gamma \in I(\mathfrak{X}, V_{le_i}) \end{array} \right\}$$

and :

$$p_{i,l} \left| \begin{array}{l} \check{\mathrm{M}}_\circ(\mathsf{v},\mathsf{w})_{i,l} \to \mathrm{M}_\circ(\mathsf{v}-le_i,\mathsf{w})_{i,0} \times \mathrm{M}_\circ(le_i,0) \\ (x,f,g,\mathfrak{X},\beta,\gamma) \mapsto (\beta_*(xf,g)_{\mathfrak{I}_i(x,f,g)}, \gamma_*(x,f,g)_{\mathfrak{X}}). \end{array} \right.$$

We study the fibers of $p_{i,l}$: take $(x,f,g) \in \mathrm{M}_\circ(\mathsf{v}-le_i,\mathsf{w})_{i,0}$ and $(z,0,0) \in \mathrm{M}_\circ(le_i,0)$ and consider $\mathfrak{I}$ and $\mathfrak{X}$ two supplementary I-graded subspaces of V_{v} such that $\dim \mathfrak{X} = le_i$, together with two isomorphisms :

$$\beta \in I(\mathfrak{I}, V_{\mathsf{v}-le_i}) \text{ and } \gamma \in I(\mathfrak{X}, V_{le_i}).$$

We identify (x,f,g) and z with $\beta^*(x,f,g)$ and $\gamma^* z$, and we define a preimage (X,F,G) by setting $(X,F,G)^{|\mathfrak{I}\oplus W}_{|\mathfrak{I}\oplus W} = (x,f,g)$, $X^{|\mathfrak{X}}_{|\mathfrak{X}} = z$ and :

$$(X,F)^{|\mathfrak{I}\oplus W}_{|\mathfrak{X}} = (\eta,\theta) \in \bar{E}(\mathfrak{X},\mathfrak{I}) \oplus \mathrm{Hom}(\mathfrak{X}_i, W_i).$$

In order to get $\mu_{\mathsf{v},\mathsf{w}}(X,F,G) = 0$, (η,θ) must satisfy the following relation :

$$\psi(\eta,\theta) = \sum_{h\in H: s(h)=i} \epsilon(h)(y_{\bar{h}}\eta_h + \eta_{\bar{h}} z_h) + g_i\theta_i = 0.$$

We need to show that ψ is surjective to conclude. Consider $\xi \in \mathrm{Hom}(\mathfrak{I}_i, \mathfrak{X}_i)$ such that $\mathrm{Tr}(\psi(\eta, \theta)\xi) = 0$ for every (η, θ). Then we have for every edge $h \in H$ such that $s(h) = i \neq j = t(h)$ and for every η_h :

$$\mathrm{Tr}(x_{\bar{h}}\eta_h\xi) = 0,$$

where the member of the left is equal to $\mathrm{Tr}(\eta_h\xi x_{\bar{h}})$. Hence $\xi x_{\bar{h}} = 0$ and $\mathrm{Im}\, x_{\bar{h}} \subseteq \ker \xi$. We also have $\mathrm{Tr}(g_i\theta_i\xi) = 0$ for every θ_i, so we similarly get $\mathrm{Im}\, g_i \subseteq \ker \xi$. Now consider a loop $h \in H$ at i. We have for every η_h :

$$\mathrm{Tr}(x_{\bar{h}}\eta_h\xi) - \mathrm{Tr}(\eta_h z_{\bar{h}}\xi) = 0.$$

Here the member of the left is equal to :

$$\mathrm{Tr}(\eta_h\xi x_{\bar{h}}) - \mathrm{Tr}(\eta_h z_{\bar{h}}\xi) = \mathrm{Tr}(\eta_h(\xi x_{\bar{h}} - z_{\bar{h}}\xi)),$$

hence $\xi x_{\bar{h}} = z_{\bar{h}}\xi$ and therefore $\ker \xi$ is stable by $x_{\bar{h}}$. We finally get :

$$\mathfrak{I}_i(x, f, g) \subseteq \ker \xi \oplus (\oplus_{j \neq i} V_{\mathsf{v}_j}).$$

Since $(x, f, g) \in \mathrm{M}_\circ(\mathsf{v} - le_i, \mathsf{w})_{i,0}$, we get $\xi = 0$, which finishes the proof. □

Corollary 2.9. *We also have a bijection :*

$$\mathrm{l}_\circ(\mathsf{v}, \mathsf{w})_{i,l} : \mathrm{Irr}\, \mathrm{L}_\circ(\mathsf{v}, \mathsf{w})_{i,l} \xrightarrow{\sim} \mathrm{Irr}\, \mathrm{L}_\circ(\mathsf{v} - le_i, \mathsf{w})_{i,0} \times \mathrm{Irr}\, \mathrm{L}_\circ(le_i, 0).$$

Proof. The image of a seminilpotent element by $p_{i,l}$ is a pair of seminilpotent elements, and the fiber of $p_{i,l}$ over a pair of seminilpotent elements consists in seminilpotent elements. □

2.2 Extension to the stable locus

Notations 2.10. Consider an inclusion of vector spaces $E \subseteq V$, F any subset of E, and $(u_j)_{1 \leq j \leq r} \in (\mathrm{End}\, V)^r$. We write :

$$\langle (u_j), F \rangle = E$$

if E is the smallest (u_j)-stable subspace of V containing F.

We will often use the following well-known fact :

Lemma 2.11. *Consider $y \in \mathrm{End}\, \mathfrak{I}$ and $z \in \mathrm{End}\, \mathfrak{X}$ such that $\mathrm{Spec}\, y \cap \mathrm{Spec}\, z = \varnothing$. If $\langle y, v \rangle = \mathfrak{I}$ and $\langle z, v' \rangle = \mathfrak{X}$ for some $v \in \mathfrak{I}$ and $v' \in \mathfrak{X}$, then $\langle y \oplus z, v \oplus v' \rangle = \mathfrak{I} \oplus \mathfrak{X}$.*

Notations 2.12. Let i be imaginary and put $\Omega(i) = \{b_1, \ldots, b_{\omega_i}\}$. For every $(x, f) \in \mathrm{L}_\circ(\mathsf{v}, \mathsf{w})$, we set $\sigma_i(x) = {}^t x_{\bar{b}_1}$.

Lemma 2.13. *With the same notations, for every $C \in \mathrm{Irr}\, \Lambda(le_i)$, there exists $x \in C$ such that :*

$$\exists \nu \in {}^t V_{le_i}, \langle \sigma_i(x), \nu \rangle = {}^t V_{le_i}.$$

Proof. It's a consequence of sections 1.1 and 1.2. If $\omega_i = 1$ and λ is a partition of l, denote by μ the conjugate partition of λ. Let $x \in \mathcal{O}_\lambda$ be defined in a base :

$$e = (e_{1,1}, \dots, e_{1,\mu_1}, \dots, e_{r,1}, \dots, e_{r,\mu_r})$$

by :

$$^t x_{b_1} = \begin{pmatrix} J_{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\mu_r} \end{pmatrix} \text{ and } {}^t x_{\bar{b}_1} = \begin{pmatrix} t_1 I_{\mu_1} + J_{\mu_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t_r I_{\mu_r} + J_{\mu_r} \end{pmatrix}$$

where the t_i are all distinct and nonzero, and :

$$J_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & & 0 & 0 \end{pmatrix}.$$

It is enough to consider ν with nonzero coordinates relatively to $(e_{1,\mu_1}, \dots, e_{r,\mu_r})$ to get $\langle \sigma_i(x), \nu \rangle = {}^t V_{le_i}$. If $\omega_i \geq 2$, we use the proof of 1.7 : in any irreducible component we can define x such that there exists v such tat $\langle x_{\bar{b}_1}, v \rangle = V_{le_i}$ ($x_{\bar{b}_i}$ corresponds to y_i in the aforementioned proof, x_{b_i} to x_i). We get the result by duality. $\square$

Remark 2.14. Note that the case $\omega_i = 1$ is very well known since it corresponds to the case of the Hilbert scheme of points in the plane.

Notations 2.15. Denote by $a_{i,j}$ the number of edges of Ω such that $s(h) = i$ and $t(h) = j$, and denote by :

$$C = (\delta_{i,j} - a_{i,j} - a_{j,i})$$

the Cartan matrix of Q. Note that :

$$(\mathsf{v} - le_i, le_i) + (le_i, \mathsf{v} - le_i) = \langle le_i, (2I - C)(\mathsf{v} - le_i) \rangle.$$

Set also $H_i = \{h \in H \mid i = s(h) \neq t(h)\}$.

Definition 2.16. Set :

$$\mathrm{L}(\mathsf{w}) := \bigcup_{\mathsf{v}} \mathrm{L}(\mathsf{v}, \mathsf{w}) \subseteq \bigcup_{\mathsf{v}} \mathrm{L}_\circ(\mathsf{v}, \mathsf{w}) =: \mathrm{L}_\circ(\mathsf{w}),$$

and define $\mathrm{B}(\mathsf{w})$ as the smallest subset of $\mathrm{Irr}\, \mathrm{L}_\circ(\mathsf{w})$ containing the only element of $\mathrm{Irr}\, \mathrm{L}_\circ(0, \mathsf{w})$, and stable by the $\mathrm{l}_\circ(\mathsf{v}, \mathsf{w})_{i,l}^{-1}(-, \mathrm{Irr}\, \Lambda(le_i))$ for v, i, l such that :

- $\langle e_i, \mathsf{w} - C\mathsf{v} \rangle \geq -l$ if $i \in I^{\mathrm{re}}$,
- $\mathsf{w}_i + \sum_{h \in H_i} \mathsf{v}_{t(h)} > 0$ if $i \in I^{\mathrm{im}}$.

Lemma 2.17. *For every $i \in I^{\mathrm{im}}$, we write $\Omega(i) = \{b_{i,1}, \dots, b_{i,\omega_i}\}$. For every $C \in \mathrm{B}(\mathsf{w})$, there exists $(x, f) \in C$ such that :*

$$\begin{cases} (x, f) \text{ stable} \\ \forall i \in I^{\mathrm{im}}, \exists \phi_i \in {}^t W_i \oplus (\oplus_{h \in H_i} {}^t V_{\mathsf{v}_{t(h)}}), \langle \sigma_i(x), \Sigma_i(x, f)(\phi_i) \rangle = {}^t V_{\mathsf{v}_i} \end{cases} \tag{2.18}$$

where $\Sigma_i(x, f) = {}^t f_i + \sum_{h \in H_i} {}^t x_h$.

Proof. We proceed by induction on v, with first step consisting in the case of $C \in \mathrm{B}(\mathsf{w}) \cap \operatorname{Irr} \mathrm{L}_\circ(le_i, m)$ for some $l > 0$. If $i \notin I^{\mathrm{im}}$, we have $l \leq \mathsf{w}_i$ by definition of $\mathrm{B}(\mathsf{w})$, hence we can find $(x, f) \in C$ such that 2.18 since it's equivalent here to f injective. If $i \in I^{\mathrm{im}}$, we have $m_i > 0$ by definition of $\mathrm{B}(\mathsf{w})$, and we can use 2.13.

Now consider $C \in \mathrm{B}(\mathsf{w}) \cap \operatorname{Irr} \mathrm{L}_\circ(\mathsf{v}, \mathsf{w})_{i,l}$ for some v and $l > 0$, and set $(C_1, C_2) = \mathrm{l}_\circ(\mathsf{v}, \mathsf{W})_{i,l}(C)$. First assume that $i \notin I^{\mathrm{im}}$. Thanks to the induction hypothesis, we can pick $((x, f), z) \in C_1 \times C_2$ such that (x, f) satisfies 2.18. Following the notations used in the proof of 2.7, we build an element of C satisfying 2.18 by chosing (η, θ) such that $\theta + \sum_{h \in H_i} \eta_h$ is injective with values in a supplementary of $\operatorname{Im}(f_i + \sum_{h \in H_i} x_h)$ in $W_i \oplus \ker(\sum_{h \in H_i} x_{\bar{h}})$: it's possible since $l + \langle e_i, \mathsf{w} - C\mathsf{v} \rangle \geq 0$ by definition of $\mathrm{B}(\mathsf{w})$.

If $i \in I^{\mathrm{im}}$, take $(x, f) \in C_1$ satisfying 2.18 and $z \in C_2$ such that :

$$\begin{cases} \operatorname{Spec} x_{\bar{b}_{i,1}} \cap \operatorname{Spec} z_{\bar{b}_{i,1}} = \varnothing \\ \exists \psi \in {}^t V_{le_i}, \langle \sigma_i(z), \psi \rangle = {}^t V_{le_i}, \end{cases}$$

which is possible, thanks to 2.13. Still following the notations of the proof of 2.7, we build an element of C mapped to $((x, f), z)$ by considering (η, θ) such that :

$$\Big({}^t\theta + \sum_{h \in H_i} {}^t\eta_h\Big)(\phi_i) = \psi$$

where $\phi_i \in {}^t W_i \oplus (\oplus_{h \in H_i} {}^t V_{\mathsf{v}_{t(h)}})$ satisfies $\langle \sigma_i(x), \Sigma_i(x, f)(\phi_i) \rangle = {}^t\mathfrak{I}$ (we use the induction hypothesis), which is possible even if $\mathfrak{I} = \{0\}$ since we have ${}^t W_i \oplus (\oplus_{h \in H_i} {}^t V_{\mathsf{v}_{t(h)}}) \neq \{0\}$ by definition of $\mathrm{B}(\mathsf{w})$. Put $\eta_{b_{i,j}} = \eta_{\bar{b}_{i,j}} = 0$ for every $j \geq 2$, so that :

$$\psi_i(\eta, \theta) = 0 \Leftrightarrow x_{\bar{b}_{i,1}} \eta_{b_{i,1}} - \eta_{b_{i,1}} z_{\bar{b}_{i,1}} = \sum_{h \in H_i} \epsilon(h)(x_{\bar{h}} \eta_h + \eta_{\bar{h}} z_h).$$

Hence we can chose $\eta_{b_{i,1}}$ in order to satisfy the right hand side equation since :

$$\operatorname{Spec} x_{\bar{b}_{i,1}} \cap \operatorname{Spec} z_{\bar{b}_{i,1}} = \varnothing \Rightarrow (\eta_{b_{i,1}} \mapsto x_{\bar{b}_{i,1}} \eta_{b_{i,1}} - \eta_{b_{i,1}} z_{\bar{b}_{i,1}}) \text{ invertible.}$$

Thanks to 2.11, $(X, F) \in C$ satisfies :

$$\langle \sigma_i(X), \Sigma_i(X, F)(\phi_i) \rangle = {}^t V_{\mathsf{v}_i}.$$

We finally have to check the stability of (X, F) to conclude. Consider $S \subseteq \ker F$ stable by X. We have $S \cap \mathfrak{I} = \{0\}$ by stability of (x, f), thus $S \simeq S_i$ and we see S as a subspace of $\ker F \cap (\cap_{h \in H_i} \ker X_h)$. But then ${}^t S$ is stable by $\sigma_i(X)$ and contains $\operatorname{Im} {}^t F + \sum_{h \in H_i} \operatorname{Im} {}^t X_h$, and thus ϕ_i. Hence ${}^t S = V_{\mathsf{v}_i}$, and $S = \{0\}$. □

Proposition 2.19. *We have* $\mathrm{B}(\mathsf{w}) = \operatorname{Irr} \mathrm{L}(\mathsf{w})$.

Proof. Thanks to 2.17, we have $\mathrm{B}(\mathsf{w}) \subseteq \operatorname{Irr} \mathrm{L}(\mathsf{w})$. Consider $C \in \operatorname{Irr} \mathrm{L}(\mathsf{v}, \mathsf{w})_{i,l} \setminus \mathrm{B}(\mathsf{w})$ for some $l > 0$. We know (*c.f.* [Nak98, 4.6]) that if $i \in I^{\mathrm{re}}$, we necessarily have $l + \langle e_i, \mathsf{v} - C\mathsf{w} \rangle \geq 0$, and thus, by definition of $\mathrm{B}(\mathsf{w})$:

$$\mathrm{l}_\circ(\mathsf{v}, \mathsf{w})_{i,l}(C) \in \Big(\operatorname{Irr} \mathrm{L}(\mathsf{v} - le_i, \mathsf{w}) \setminus \mathrm{B}(\mathsf{w})\Big) \times \operatorname{Irr} \Lambda(le_i).$$

If $i \in I^{\mathrm{im}}$, $C \in \operatorname{Irr} \mathrm{L}(\mathsf{v}, \mathsf{w})_{i,l}$ necessarily implies $\mathsf{w}_i + \sum_{h \in H_i} \mathsf{v}_{t(h)} > 0$, and we get to the same conclusion. By descending induction on v, we obtain that the only irreducible component of $\mathrm{L}(0, \mathsf{w})$ doesn't belong to $\mathrm{B}(\mathsf{w})$, which is absurd. □

Corollary 2.20. *Take $i \in I^{\mathrm{im}}$ and assume* $\mathrm{Irr}\,\mathsf{L}(\mathsf{v},\mathsf{w})_{i,l} \subseteq \mathsf{B}(\mathsf{w})$. *We have the following commutative diagram :*

$$\begin{array}{ccc}
\mathrm{Irr}\,\mathsf{L}(\mathsf{v},\mathsf{w})_{i,l} & \xrightarrow[\sim]{\mathfrak{l}(\mathsf{v},\mathsf{w})_{i,l}} & \mathrm{Irr}\,\mathsf{L}(\mathsf{v}-le_i,\mathsf{w})_{i,0} \times \mathrm{Irr}\,\Lambda(le_i) \\
\sim\downarrow & & \downarrow\sim \\
\mathrm{Irr}\,\mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l} & \xrightarrow[\sim]{\mathfrak{l}(\mathsf{v},\mathsf{w})_{i,l}} & \mathrm{Irr}\,\mathfrak{L}(\mathsf{v}-le_i,\mathsf{w})_{i,0} \times \mathrm{Irr}\,\Lambda(le_i).
\end{array} \tag{2.21}$$

Proof. By definition of stability, the action of G_{v} on $\mathsf{L}(\mathsf{v},\mathsf{w})$ is free. □

Theorem 2.22. *The subvariety $\mathfrak{L}(\mathsf{v},\mathsf{w}) \subset \mathfrak{M}(\mathsf{v},\mathsf{w})$ is Lagrangian.*

Proof. Thanks to 1.2, we just have to prove that every irreducible component of $\mathfrak{L}(n,m)$ is of dimension $\frac{1}{2}\langle n, 2m - Cn\rangle$. We proceed by induction, thanks to 2.20. Take $C \in \mathrm{Irr}\,\mathfrak{L}(\mathsf{v},\mathsf{w})$, thanks to 2.6, there exist $i \in I$ and $l \geq 1$ such that $C \cap \mathfrak{L}(\mathsf{v},\mathsf{w})_{i,l}$ is dense in C. Consider $(C_1, C_2) = \mathfrak{l}(\mathsf{v},\mathsf{w})_{i,l}(C)$. Thanks to 2.8 and 2.21, we have :

$$\dim C + \dim G_{\mathsf{v}} + d_q = (\dim C_1 + \dim C_2) + \dim G_{\mathsf{v}-le_i} + d_p,$$

where d_p and d_q denote the dimensions of the fibers of $p_{i,l}$ and $q_{i,l}$ in 2.8. Moreover, we know from sections 1.1 and 1.2 that :

$$\dim C_2 = \dim \Lambda(le_i) = \omega_i l^2 = \frac{1}{2}\langle le_i, (2I - C)(le_i)\rangle.$$

Hence, we get :

$$\begin{aligned}
&\dim C + 2\langle \mathsf{v},\mathsf{v}\rangle - \langle le_i, \mathsf{v} - le_i\rangle \\
&\qquad = \dim C_1 + \frac{1}{2}\langle le_i, (2I-C)(le_i)\rangle + \langle \mathsf{v} - le_i, \mathsf{v} - le_i\rangle \\
&\qquad\qquad + \langle le_i, (2I - C)(\mathsf{v} - le_i)\rangle + \langle \mathsf{v},\mathsf{v}\rangle - \langle le_i, \mathsf{v} - \mathsf{w} - le_i\rangle.
\end{aligned}$$

As $\mathfrak{L}(\mathsf{v} - le_i, \mathsf{w})_{i,0}$ is open in $\mathfrak{L}(\mathsf{v} - le_i, \mathsf{w})$, we can use the induction hypothesis to write :

$$\begin{aligned}
\dim C &= \frac{1}{2}\langle \mathsf{v} - le_i, 2\mathsf{w} - C(\mathsf{v} - le_i)\rangle + \frac{1}{2}\langle le_i, (2I - C)(le_i)\rangle - \langle le_i, \mathsf{v}\rangle \\
&\quad + \langle le_i, \mathsf{w} + (I - C)(\mathsf{v} - le_i)\rangle \\
&= \frac{1}{2}\langle \mathsf{v} - le_i, 2\mathsf{w} - C(\mathsf{v} - le_i)\rangle - \frac{1}{2}\langle le_i, C(le_i)\rangle + \langle le_i, \mathsf{w} - C(\mathsf{v} - le_i)\rangle \\
&= \frac{1}{2}\langle \mathsf{v} - le_i, 2\mathsf{w} - C(\mathsf{v} - le_i)\rangle + \frac{1}{2}\langle le_i, C(le_i)\rangle \\
&\quad + \frac{1}{2}\langle le_i, 2\mathsf{w} - C\mathsf{v}\rangle - \frac{1}{2}\langle \mathsf{v}, C(le_i)\rangle \qquad \text{[since } C \text{ is symmetric]} \\
&= \frac{1}{2}\langle \mathsf{v}, 2\mathsf{w} - C\mathsf{v}\rangle
\end{aligned}$$

which ends the proof. □

2.3 Tensor product on Irr $\mathfrak{L}$

2.3.1 Another Lagrangian subvariety

Embed W in a $\mathsf{w}+\mathsf{w}'$-dimensional I-graded vector space, and fix a supplementary subspace W' of W. We still denote by $I(X,Y)$ the set of I-graded isomorphisms between two I-graded spaces X and Y.

For every $\mathbf{v} \in \mathbb{N}^I$, denote by $\mathrm{Z}_\circ(\mathbf{v}) \subseteq \mathrm{M}_\circ(\mathbf{v}, \mathsf{w}+\mathsf{w}')$ of elements (x,f,g) such that there exists an I-graded subspace of $V_\mathbf{v}$ satisfying :

1. $x(V) \subseteq V$;
2. $f(V) \subseteq W$;
3. $g(W \oplus W') \subseteq V$;
4. $g(W) = \{0\}$,

and denote by $V(x,f,g)$ the larger x-stable subspace of $f^{-1}(W)$ containing $\operatorname{Im} g$. We will then denote by $\widetilde{\mathrm{Z}}_\circ(\mathbf{v}) \subset \mathrm{Z}_\circ(\mathbf{v})$ the subvariety of elements (x,f,g) such that :

$$(x,f)^{|V\times W}_{|V\times V} \text{ and } (x,f)^{|(V_\mathbf{v}/V)\times(W\oplus W'/W)}_{|(V_\mathbf{v}/V)\times(V_\mathbf{v}/V)} \text{ are seminilpotents}$$

where we have written V instead of $V(x,f,g)$. We get a stratification of $\widetilde{\mathrm{Z}}_\circ(\mathbf{v})$ by setting, for any v,v' such that $\mathsf{v}+\mathsf{v}' = \mathbf{v}$:

$$\widetilde{\mathrm{Z}}_\circ(\mathsf{v},\mathsf{v}') = \left\{ (x,f,g) \in \widetilde{\mathrm{Z}}_\circ(\mathsf{v}+\mathsf{v}') \mid \dim V(x,f,g) = \mathsf{v} \right\}.$$

Define the following incidence variety :

$$\check{\mathrm{Z}}_\circ(\mathsf{v},\mathsf{v}') = \left\{ (x,f,g,V',\alpha) \;\middle|\; \begin{array}{l} (x,f,g) \in \widetilde{\mathrm{Z}}_\circ(\mathsf{v},\mathsf{v}') \\ V(x,f,g) \oplus V' = V_{\mathsf{v}+\mathsf{v}'} \\ \alpha \in I(V(x,f,g), V_\mathsf{v}) \times I(V', V_{\mathsf{v}'}) \end{array} \right\}.$$

By definition of $V(x,f,g)$ (again denoted by V hereunder), we have :

$$(x,f,g) \in \mathrm{Z}_\circ(\mathbf{v}) \Rightarrow (x,f)^{|(V_\mathbf{v}/V)\times(W\oplus W'/W)}_{|(V_\mathbf{v}/V)\times(V_\mathbf{v}/V)} \text{ stable,}$$

hence the following application is well defined :

$$\mathrm{T}_\circ \left| \begin{array}{l} \check{\mathrm{Z}}_\circ(\mathsf{v},\mathsf{v}') \to \mathrm{L}_\circ(\mathsf{v},\mathsf{w}) \times \mathrm{L}(\mathsf{v}',\mathsf{w}') \\ (x,f,g,V',\alpha) \mapsto (\alpha\times\beta)_* \left((x,f)^{|V\times W}_{|V\times V}, (x,f)^{|V'\times(W\oplus W'/W)}_{|V'\times V'} \right) \end{array} \right.$$

Proposition 2.23. *The map* $\mathrm{T}_\circ$ *is smooth with connected fibers of dimension :*

$$\langle \mathsf{v}+\mathsf{v}', \mathsf{v}+\mathsf{v}' \rangle + \langle \mathsf{w}', \mathsf{v} \rangle - \langle \mathsf{v}, C\mathsf{v}' \rangle + \langle \mathsf{v}', \mathsf{w} \rangle + \langle \mathsf{v}, \mathsf{v}' \rangle.$$

Proof. Let (x,f) and (x',f') be elements of $\mathrm{L}_\circ(\mathsf{v},\mathsf{w})$ and $\mathrm{L}(\mathsf{v}',\mathsf{w}')$ and take I-graded spaces V and V' of dimensions v and v'. Define (X,F,G,V',α) in the fiber $\mathrm{T}_\circ^{-1}((x,f),(x',f'))$ by :

1. $\alpha \in I(V, V_\mathsf{v}) \times I(V', V_{\mathsf{v}'})$;

2. $G = 0 \oplus \nu$ where :
$$\nu \in \oplus_{i\in I} \operatorname{Hom}(W'_i, V_i);$$
3. $X = \alpha^* x \oplus (\alpha^* x' + \eta) : V \oplus V' \to V \oplus V'$ where :
$$\eta \in \oplus_{h\in H} \operatorname{Hom}(V'_{s(h)}, V_{t(h)});$$
4. $F = \alpha^* f \oplus (\alpha^* f' + \theta) : V \oplus V' \to W \oplus W'$ where :
$$\theta \in \oplus_{i\in I} \operatorname{Hom}(V'_i, W_i);$$

such that $\mu_{\mathsf{v}+\mathsf{v}',\mathsf{w}+\mathsf{w}'}(X, F, G) = 0$.

Lemma 2.24. *This equation is linear in the variables (ν, η, θ), and the associated linear map is surjective.*

Proof. We first identify x, x', and f' with $\alpha^* x, \alpha^* x'$, and $\alpha^* f'$. Then the linear map $\zeta = (\zeta_i)$ we're interested in is given by :
$$\zeta_i(\nu, \eta, \theta) = \nu_i f'_i + \sum_{h\in H: s(h)=i} \epsilon(\bar{h})(x_{\bar{h}}\eta_h + \eta_{\bar{h}} x'_h).$$
Take $L \in \oplus_{i\in I} \operatorname{Hom}(V_i, V'_i)$ such that for every (ν, η, θ) :
$$\sum_{i\in I} \operatorname{Tr}(\zeta(\nu, \eta, \theta) L_i) = 0.$$
Then for every edge h such that $s(h) = i$, $t(h) = j$, we have for every η_h :
$$\operatorname{Tr}(x_{\bar{h}}\eta_h L_i) - \operatorname{Tr}(\eta_h x'_{\bar{h}} L_j) = 0.$$
But
$$\operatorname{Tr}(\eta_h L_i x_{\bar{h}}) - \operatorname{Tr}(\eta_h x'_{\bar{h}} L_j) = \operatorname{Tr}(\eta_h L_i x_{\bar{h}} - \eta_h x'_{\bar{h}} L_j) = \operatorname{Tr}(\eta_h (L_i x_{\bar{h}} - x'_{\bar{h}} L_j))$$
Hence $L_i x_{\bar{h}} = x'_{\bar{h}} L_j$, and thus $\operatorname{Im} L$ is stable by x'. Moreover :
$$\forall i, \forall \nu_i, \operatorname{Tr}(\nu_i f'_i L_i) = 0 \Rightarrow \forall i, f'_i L_i = 0 \Rightarrow \operatorname{Im} L \subset \ker f',$$
hence the lemma comes from the stability of (x', f'). □

We have to check that $V = V(X, F, G)$. It is easy to see that $V \subset V(X, F, G)$. Moreover :
$$F^{-1}(W) = \{v + v' \in V \oplus V' \mid f(v) + \theta(v') + f'(v') \in W\} = V \oplus \ker f',$$
hence, if Y is an X-stable subspace of $F^{-1}(W)$, Y/V is an x'-stable subspace of $\ker f'$. Since (x', f') is stable, we have $Y \subset V$, and thus $V = V(X, F, G)$.

We have proved that the fiber $\mathrm{T}_\circ^{-1}((x, f), (x', f'))$ is isomorphic to :
$$G_{\mathsf{v}+\mathsf{v}'} \times \mathbb{C}^{\langle \mathsf{w}', \mathsf{v}\rangle + (\mathsf{v}', \mathsf{v}) + \langle \mathsf{v}', \mathsf{w}\rangle - \langle \mathsf{v}', \mathsf{v}\rangle}$$
and thus is connected. Since $(\mathsf{v}, \mathsf{v}') = 2\langle \mathsf{v}, \mathsf{v}'\rangle - \langle \mathsf{v}, C\mathsf{v}'\rangle$, its dimension is :
$$\begin{aligned} d_{\mathrm{T}} &= \langle \mathsf{v}+\mathsf{v}', \mathsf{v}+\mathsf{v}'\rangle + \langle \mathsf{w}', \mathsf{v}\rangle + 2\langle \mathsf{v}, \mathsf{v}'\rangle - \langle \mathsf{v}, C\mathsf{v}'\rangle + \langle \mathsf{v}', \mathsf{w}\rangle - \langle \mathsf{v}', \mathsf{v}\rangle \\ &= \langle \mathsf{v}+\mathsf{v}', \mathsf{v}+\mathsf{v}'\rangle + \langle \mathsf{w}', \mathsf{v}\rangle - \langle \mathsf{v}, C\mathsf{v}'\rangle + \langle \mathsf{v}', \mathsf{w}\rangle + \langle \mathsf{v}, \mathsf{v}'\rangle. \end{aligned}$$

□

Lemma 2.25. *Consider* $(x, f, g) \in \widetilde{\mathrm{Z}}_\circ(\mathsf{v}, \mathsf{v}')$ *and* $V = V(x, f, g)$. *Then :*

$$(x, f, g) \text{ stable } \Leftrightarrow (x, f)_{|V\times V}^{|V\times W} \text{ stable}$$

and we denote by $\widetilde{\mathrm{Z}}(\mathsf{v}, \mathsf{v}')$ *the subvariety of stable points of* $\widetilde{\mathrm{Z}}_\circ(\mathsf{v}, \mathsf{v}')$*, and :*

$$\widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') = \widetilde{\mathrm{Z}}(\mathsf{v}, \mathsf{v}')/\!\!/G_{\mathsf{v}+\mathsf{v}'}.$$

Proof. The equivalence is a consequence of the following facts :
- the restriction of a stable point is stable ;
- the extension of a stable point by a stable point is stable ;
- the point $(x, f)_{|(V_{\mathsf{v}+\mathsf{v}'}/V)\times(V_{\mathsf{v}+\mathsf{v}'}/V)}^{|(V_{\mathsf{v}+\mathsf{v}'}/V)\times(W\oplus W'/W)}$ is stable.

□

Theorem 2.26. *We have the following bijection :*

$$\mathrm{Irr}\, \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \mathrm{Irr}\, \mathfrak{L}(\mathsf{v}, \mathsf{w}') \xrightarrow[\sim]{\otimes} \mathrm{Irr}\, \widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}').$$

Moreover, the subvariety $\widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') \subset \mathfrak{M}(\mathsf{v}+\mathsf{v}', \mathsf{w}+\mathsf{w}')$ *is Lagrangian.*

Proof. Define $\check{\mathrm{Z}}(\mathsf{v}, \mathsf{v}')$ as the variety of stable points of $\check{\mathrm{Z}}_\circ(\mathsf{v}, \mathsf{v}')$. We have the following diagram :

$$\begin{tikzcd}
\check{\mathrm{Z}}(\mathsf{v}, \mathsf{v}') \arrow[r, "\mathrm{T}"] \arrow[d] & \mathrm{L}(\mathsf{v}, \mathsf{w}) \times \mathrm{L}(\mathsf{v}', \mathsf{w}') \arrow[d] \\
\widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') \arrow[r, "\mathfrak{T}"] & \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \mathfrak{L}(\mathsf{v}', \mathsf{w}')
\end{tikzcd}$$

where the rightmost vertical map is juste the free quotient by $G_{\mathsf{v}} \times G_{\mathsf{v}'}$. The leftmost map being a principal bundle with fibers isomorphic to :

$$G_{\mathsf{v}} \times G_{\mathsf{v}'} \times \mathrm{Grass}^I_{\mathsf{v},\mathsf{v}'}(\mathsf{v}+\mathsf{v}') \times G_{\mathsf{v}+\mathsf{v}'},$$

we get our bijection thanks to 2.23 and 2.25. Moreover :

$$\begin{aligned}
\dim \widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') &+ \dim\{G_{\mathsf{v}} \times G_{\mathsf{v}'} \times \mathrm{Grass}^I_{\mathsf{v},\mathsf{v}'}(\mathsf{v}+\mathsf{v}') \times G_{\mathsf{v}+\mathsf{v}'}\} \\
&= \dim\{\mathfrak{L}(n, m) \times \mathfrak{L}(n', m')\} + \dim G_n + \dim G_{n'} + d_{\mathrm{T}}
\end{aligned}$$

where d_{T} is the dimension of the fibers of T. Thanks to 2.25, this dimension is the same as the dimension of the fibers of $\mathrm{T}_\circ$. Hence, by 2.23 and 2.22 :

$$\begin{aligned}
\dim \widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') &+ \langle \mathsf{v}, \mathsf{v}\rangle + \langle \mathsf{v}', \mathsf{v}'\rangle + \langle \mathsf{v}, \mathsf{v}'\rangle + \langle \mathsf{v}+\mathsf{v}', \mathsf{v}+\mathsf{v}'\rangle \\
&= \frac{1}{2}\langle \mathsf{v}, 2\mathsf{w} - C\mathsf{v}\rangle + \frac{1}{2}\langle \mathsf{v}', 2\mathsf{w}' - C\mathsf{v}'\rangle + \langle \mathsf{v}, \mathsf{v}\rangle + \langle \mathsf{v}', \mathsf{v}'\rangle \\
&\quad + \langle \mathsf{v}+\mathsf{v}', \mathsf{v}+\mathsf{v}'\rangle + \langle \mathsf{w}', \mathsf{v}\rangle - \langle \mathsf{v}, C\mathsf{v}'\rangle + \langle \mathsf{v}', \mathsf{w}\rangle + \langle \mathsf{v}, \mathsf{v}'\rangle
\end{aligned}$$

Finally (*c.f.* C symmetric) :

$$\dim \widetilde{\mathfrak{Z}}(\mathsf{v}, \mathsf{v}') = \frac{1}{2}\langle \mathsf{v}+\mathsf{v}', 2(\mathsf{w}+\mathsf{w}') - C(\mathsf{v}+\mathsf{v}')\rangle.$$

The Lagrangian character is now a consequence of 2.4. □

The results of the section 2.2 can be adapted to prove the following result (the case $\omega_i = 0$ being treated in [Nak01, §4]) :

Proposition 2.27. *Consider i such that $\omega_i > 0$ and $l > 0$. If :*

$$\mathsf{w} + \mathsf{w}' + \sum_{h \in H_i} \mathbf{v}_{t(h)} > 0,$$

we have a bijection :

$$\operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l} \xrightarrow{\sim} \operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{v} - le_i)_{i,0} \times \operatorname{Irr} \Lambda(le_i).$$

2.3.2 Comparison of two crystal-type structures

Notations 2.28. For every $X \in \operatorname{Irr} \widetilde{\mathfrak{Z}}(\mathbf{v})_{i,l}$, we will denote by $\epsilon_i(X) \in \operatorname{Irr} \Lambda(le_i)$ the composition of the second projection with the bijection obtained in 2.27, and $|\epsilon_i(X)| = l$. Note that if $(X, X') \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \operatorname{Irr} \mathfrak{L}(\mathsf{v}', \mathsf{w}')$, the quantity $\epsilon_i(X \otimes X')$ makes sense thanks to 2.26 and 2.27.

We will write $\Omega(i) = \{b_{i,j}\}_{1 \leq j \leq \omega_i}$ for i imaginary, or $\Omega(i) = \{b_j\}_{1 \leq j \leq \omega_i}$ if it is not ambiguous.

Lemma 2.29. *Let i be an imaginary vertex and assume $\sum_{h \in H_i} n_{t(h)} > 0$. For every $C \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})$, there exists $(x, f) \in C$, $v \in \operatorname{Im} \sum_{h \in H_i} x_{\bar{h}}$ such that :*

$$\langle x_{\bar{b}_1}, v \rangle = \mathfrak{I}_i(x, f).$$

Proof. We proceed by induction on v_i, the first step being trivial. For the inductive step, we can immediatly conclude if $C \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,l}$ for $l > 0$. Otherwise, $C \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,0}$, but $C \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{j,l}$ for some $j \in I$ and $l > 0$. There exists a minimal chain $(j_k, l_k, C_k)_{1 \leq k \leq s}$ of elements of $I \times \mathbb{N}_{>0} \times \operatorname{Irr} \mathfrak{L}(-, \mathsf{w})$ such that :
- $(j_1, l_1, C_1) = (j, l, C)$;
- $C_{k+1} = \mathrm{pr}_1 \mathfrak{l}(\mathsf{v} - l_1 j_1 - \cdots - l_k j_k, \mathsf{w})_{j_k, l_k}(C_k)$ where pr_1 is the first projection ;
- $j_s = i$.

We necessarely have j_{s-1} adjacent to i, and by induction hypothesis, the proposition is satisfied by C_s, and thus by C_{s-1}. But then, thanks to 2.11 and 2.13, the proposition is also satisfied by C_{s-2} for a generic choice of $\eta_{\bar{h}}$ (using the notations of the proof of 2.17 where i is replaced by j_{s-1}). Hence it is also satisfied by $C = C_1$. □

Proposition 2.30. *Let i be an imaginary vertex and consider $(X, X') \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \operatorname{Irr} \mathfrak{L}(\mathsf{v}', \mathsf{w}')$. Assume $|\epsilon_i(X')| < \mathsf{v}'_i$ or $0 < \mathsf{w}'_i$. Then we have :*

$$\epsilon_i(X \otimes X') = \epsilon_i(X').$$

Proof. Put $(Y, C) = \mathfrak{l}(n, m)_{i,l}(X)$ where $l = |\epsilon_i(X)|$. Take $((x, f), (x', f')) \in X \times X'$. Consider the equation $\zeta_i = 0$ used in the proof of 2.24 :

$$\nu_i f'_i + \sum_{h \in H : s(h) = i} \epsilon(\bar{h})(x_{\bar{h}} \eta_h + \eta_{\bar{h}} x'_h) = 0.$$

Note $\eta_{b_j} = \eta_j$, $x_{b_j} = x_j$ and $x_{\bar{b}_j} = \bar{x}_j$ (and the same with x'), take $\eta_{\bar{b}_j} = 0$ so that our equation becomes :

$$\begin{aligned}\nu_i f'_i + \sum_{h \in H_i} \eta_{\bar{h}} x'_h &= \sum_{1 \le j \le \omega_i} (\bar{x}_j \eta_j - \eta_j \bar{x}'_j) \\ &= \bar{x}_1 \eta_1 - \eta_1 \bar{x}'_1\end{aligned}$$

if we also set $\eta_j = 0$ for $j \ge 2$ (if any). Then, we set :

$$\begin{aligned}x' &= f'_i + \bigoplus_{h \in H_i} x'_h : V_{\mathsf{v}'_i} \to W'_i \oplus \bigoplus_{h \in H_i} V_{\mathsf{v}'_{t(h)}} \\ \bar{\eta} &= \nu_i + \sum_{h \in H_i} \epsilon(\bar{h}) \eta_{\bar{h}} : W'_i \oplus \bigoplus_{h \in H_i} V_{\mathsf{v}'_{t(h)}} \to V_{\mathsf{v}_i} \\ \bar{x} &= \sum_{h \in H_i} \epsilon(\bar{h}) x_{\bar{h}} : \bigoplus_{h \in H_i} V_{\mathsf{v}_{t(h)}} \to V_{\mathsf{v}_i} \\ \eta &= \bigoplus_{h \in H_i} \eta_h : V_{\mathsf{v}'_i} \to \bigoplus_{h \in H_i} V_{\mathsf{v}_{t(h)}}\end{aligned}$$

and our equation finally becomes :

$$\bar{\eta} x' + \eta \bar{x} = \bar{x}_1 \eta_1 - \eta_1 \bar{x}'_1.$$

Consider the open subvariety of $X \times X'$ where :

1. there exists $\mathbf{v} \in V_{\mathsf{v}_i}$ such that its image $\bar{\mathbf{v}} \in V_{\mathsf{v}_i} / \mathfrak{I}_i(x, f)$ satisfies :
$$\langle \bar{x}_{1 | V_{\mathsf{v}_i} / \mathfrak{I}_i(x,f)}, \bar{\mathbf{v}} \rangle = V_{\mathsf{v}_i} / \mathfrak{I}_i(x, f);$$
2. $\bar{x}'_1$, $\bar{x}_{1 | \mathfrak{I}_i(x,f)}$ and $\bar{x}_{1 | \mathbb{C}^{n_i} / \mathfrak{I}_i(x,f)}$ have disjoint spectra ;
3. there exist v and v' such that $\mathbf{w} = \sum_{h \in H_i} x_{\bar{h}}(v)$ and $\mathbf{w}' = \sum_{h \in H_i} x'_{\bar{h}}(v')$ satisfy :
$$\langle \bar{x}_1 \oplus \bar{x}'_1, \mathbf{w} \oplus \mathbf{w}' \rangle = \mathfrak{I}_i(x, f) \oplus \mathfrak{I}_i(x', f');$$

which is nonempty, thanks to 2.13, 2.29 and 2.11. Take :

- $\bar{\eta} = \nu_i$ and $\mathbf{v} \in \operatorname{Im} \nu_i$ if $\mathsf{w}'_i > 0$;
- $\bar{\eta}$ such that $\bar{\eta}(v') = \mathbf{v}$ if $\mathsf{v}'_i > |\epsilon_i(X')|$ (possible since $v' \neq 0$).

From 2.11, we get (with the notations used in the proof of 2.23) :

$$\left\langle X_{\bar{b}_1}, \operatorname{Im}\Big(\sum_{h \in H_i} X_{\bar{h}} \Big) \right\rangle = V_{\mathsf{v}_i} \oplus \mathfrak{I}_i(x', f').$$

We have to check that we can chose η such that the equations $\zeta_{t(h)} = 0$ are satisfied for every $h \in H_i$ (if $\mathsf{w}'_i > 0$ and $\bar{\eta} = \nu_i$, just take $\eta = 0$). It suffices to set $\eta_h x'_{\bar{h}}(v'_{t(h)}) = -x_h \eta_{\bar{h}}(v'_{t(h)})$ (possible since $\mathsf{v}'_i > |\epsilon_i(X')|$ and since we may assume that $v'_{t(h)} = 0$ if $x'_{\bar{h}}(v'_{t(h)}) = 0$) and to set η and $\bar{\eta}$ equal to zero on supplementaries of $\mathbb{C}\mathbf{w}'$ and $\mathbb{C}v'$ respectively. We can finally chose η_1 such that $\bar{\eta} x' + \eta \bar{x} = \bar{x}_1 \eta_1 - \eta_1 \bar{x}'_1$ (possible since $\operatorname{Spec} \bar{x}'_1 \cap \operatorname{Spec} \bar{x}_1 = \varnothing$). Since :

$$\operatorname{codim} \mathfrak{I}_i(x, f) \ge |\epsilon_i(X')|,$$

for every $(x, f) \in X \otimes X'$, the subvariety of $X \otimes X'$ defined by :

$$\operatorname{codim} \mathfrak{I}_i(x, f) = |\epsilon_i(X')|,$$

is open, and we have shown it is non empty, hence the theorem is proved. □

Proposition 2.31. *Assume* $\mathsf{w}'_i = 0$, $|\epsilon_i(X')| = \mathsf{v}'_i$ *and* $\sum_{h\in H_i} \mathsf{v}'_{t(h)} > 0$. *Then we still have* $\epsilon_i(X \otimes X') = \epsilon_i(X')$.

Proof. Thanks to the previous proof, the result is clear if there exists an imaginary vertex j adjacent to i : the choice of $x_{\bar{b}_{j,1}}$ and $x'_{\bar{b}_{j,1}}$ with disjoint spectra enables to use $\eta_{b_{j,1}}$ for $\zeta_j = 0$ to be satisfied (with the usual notation $\Omega(j) = \{b_{j,1}, \dots, b_{j,\omega_j}\}$).

Assume that every neighbour of i is real. Following the previous proof, assume $\bar{\eta} = \eta_{\bar{h}}$ is of rank 1 for some $h : i \to j$. We have to check that $\zeta_j = 0$ can be satisfied. It is clear if $f'_j \neq 0$: just chose ν_j such that $\nu_j f'_j = -\epsilon(h) x_h \eta_{\bar{h}}$ and $\eta_p = 0 = \eta_{\bar{p}}$ if $p \in H_j \setminus \{\bar{h}\}$, so that $\zeta_j = 0$ is satisfied. Otherwise, there necessarily exists an edge $q : j \to k \neq i$ such that $x'_q \neq 0$ (if not, $V'_{\mathsf{v}'_i} \oplus V'_{\mathsf{v}'_j} \subseteq \ker f'$ would be x'-stable, which is not possible for every vertex j adjacent to i since $\sum_{h\in H_i} \mathsf{v}'_{t(h)} > 0$). Hence it is possible to chose $\eta_{\bar{q}}$ so that $\epsilon(\bar{q})\eta_{\bar{q}} x'_q = -\epsilon(h) x_h \eta_{\bar{h}}$ and $\eta_p = 0 = \eta_{\bar{p}}$ if $p \in H_j \setminus \{\bar{h}, q\}$, and thus get $\zeta_j = 0$ satisfied. □

We have proved the following :

Theorem 2.32. *Let i be an imaginary vertex and consider* $(X, X') \in \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w}) \times \operatorname{Irr} \mathfrak{L}(\mathsf{v}', \mathsf{w}')$. *We have :*

$$\epsilon_i(X \otimes X') = \begin{cases} \epsilon_i(X') & \text{if } \mathsf{w}'_i + \sum_{h\in H_i} \mathsf{v}'_{t(h)} > 0 \\ \epsilon_i(X) & \text{otherwise.} \end{cases}$$

3 A generalization of crystals

Notations 3.1. Put $\mathcal{C}_{i,l} = \operatorname{Irr} \Lambda(l e_i)$ *i.e.* the singleton $\{l\}$ if $i \in I^{\text{re}}$, the set of partitions if $\omega_i = 1$ (denoted by $\lambda = (\lambda_1 \leq \dots \leq \lambda_r)$), the set of compositions otherwise (denoted by $\mathsf{c} = (\mathsf{c}_1, \dots, \mathsf{c}_r)$), and set $\mathcal{C}_i = \sqcup_{l\geq 0} \mathcal{C}_{i,l}$. If $\mathsf{c} \in \mathcal{C}_i$, we write $\mathsf{c} \setminus \mathsf{c}_1$ for $(\mathsf{c}_2, \dots, \mathsf{c}_r)$. Denote by P the free $\mathbb{Z}$-lattice spanned by the family $(e_i)_{i\in I}$. We will also note α_i instead of Ce_i.

Definition 3.2. We call Q-crystal a set $\mathcal{B}$ together with maps :

$$\begin{aligned} \mathrm{wt} &: \mathcal{B} \to P \\ \epsilon_i &: \mathcal{B} \to \mathcal{C}_i \\ \phi_i &: \mathcal{B} \to \mathbb{N} \sqcup \{+\infty\} \\ \tilde{e}_i, \tilde{f}_i &: \mathcal{B} \to \mathcal{B} \sqcup \{0\} && i \in I^{\text{re}} \\ \tilde{e}_{i,l}, \tilde{f}_{i,l} &: \mathcal{B} \to \mathcal{B} \sqcup \{0\} && i \in I^{\text{im}}, l > 0 \end{aligned}$$

such that for every $b, b' \in \mathcal{B}$:

1. $\langle e_i, \mathrm{wt}(b) \rangle \geq 0$ if $i \in I^{\text{im}}$;
2. $\mathrm{wt}(\tilde{e}_{i,l} b) = \mathrm{wt}(b) + l\alpha_i$ if $\tilde{e}_{i,l} b \neq 0$;
3. $\mathrm{wt}(\tilde{f}_{i,l} b) = \mathrm{wt}(b) - l\alpha_i$ if $\tilde{f}_{i,l} b \neq 0$;
4. $\tilde{f}_{i,l} b = b' \Leftrightarrow b = \tilde{e}_{i,l} b'$;

5. if $\tilde{e}_{i,l}b \neq 0$, $\epsilon_i(\tilde{e}_{i,l}b) = \begin{cases} \epsilon_i(b) - l & \text{if } i \in I^{\text{re}} \\ \epsilon_i(b) \setminus l & \text{if } i \in I^{\text{im}} \text{ and } l = \epsilon_i(b)_1 \\ 0 & \text{otherwise}; \end{cases}$

6. if $\tilde{f}_{i,l}b \neq 0$, $\epsilon_i(\tilde{f}_{i,l}b) = \begin{cases} \epsilon_i(b) + l & \text{if } i \in I^{\text{re}} \\ (l, \epsilon_i(b)) & \text{if } \omega_i \geq 2 \text{ or } \omega_i = 1 \text{ and } l \leq \epsilon_i(b)_1 \\ 0 & \text{otherwise}; \end{cases}$

7. $\phi_i(b) = \begin{cases} \epsilon_i(b) + \langle e_i, \text{wt}(b) \rangle & \text{if } i \in I^{\text{re}} \\ +\infty & \text{if } i \in I^{\text{im}} \text{ and } \langle e_i, \text{wt}(b) \rangle > 0 \\ 0 & \text{otherwise}, \end{cases}$

where, for $i \in I^{\text{re}}$, we write $\tilde{e}_{i,1}, \tilde{f}_{i,1}$ instead of $\tilde{e}_i, \tilde{f}_i$ and $\tilde{e}_{i,l}, \tilde{f}_{i,l}$ instead of $\tilde{e}_{i,1}^l, \tilde{f}_{i,1}^l$.

Remark 3.3. Note that this definition of ϕ_i already appears in [JKK05].

Definition 3.4. The *tensor product* $\mathcal{B} \otimes \mathcal{B}' = \{b \otimes b' \mid b \in \mathcal{B}, b' \in \mathcal{B}'\}$ of two crystals is defined by :

1. $\text{wt}(b \otimes b') = \text{wt}(b) + \text{wt}(b')$;
2. if $i \in I^{\text{re}}$, $\epsilon_i(b \otimes b') = \max\{\epsilon_i(b), \epsilon_i(b') - \langle e_i, \text{wt}(b) \rangle\}$;
3. if $i \in I^{\text{im}}$, $\epsilon_i(b \otimes b') = \begin{cases} \epsilon_i(b) & \text{if } \phi_i(b) = +\infty \\ \epsilon_i(b') & \text{if } \phi_i(b) = 0; \end{cases}$
4. if $i \in I^{\text{re}}$, $\phi_i(b \otimes b') = \max\{\phi_i(b) + \langle e_i, \text{wt}(b') \rangle, \phi_i(b')\}$;
5. if $i \in I^{\text{im}}$, $\phi_i(b \otimes b') = \begin{cases} \phi_i(b) & \text{if } \phi_i(b) = +\infty \\ \phi_i(b') & \text{if } \phi_i(b) = 0; \end{cases}$
6. if $i \in I^{\text{re}}$, $\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i(b) \otimes b' & \text{if } \phi_i(b) \geq \epsilon_i(b') \\ b \otimes \tilde{e}_i(b') & \text{if } \phi_i(b) < \epsilon_i(b'); \end{cases}$
7. if $i \in I^{\text{re}}$, $\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i(b) \otimes b' & \text{if } \phi_i(b) > \epsilon_i(b') \\ b \otimes \tilde{f}_i(b') & \text{if } \phi_i(b) \leq \epsilon_i(b'); \end{cases}$
8. if $i \in I^{\text{im}}$, $\tilde{e}_{i,l}(b \otimes b') = \begin{cases} \tilde{e}_{i,l}(b) \otimes b' & \text{if } \phi_i(b) = +\infty \\ b \otimes \tilde{e}_{i,l}(b') & \text{if } \phi_i(b) = 0; \end{cases}$
9. if $i \in I^{\text{im}}$, $\tilde{f}_{i,l}(b \otimes b') = \begin{cases} \tilde{f}_{i,l}(b) \otimes b' & \text{if } \phi_i(b) = +\infty \\ b \otimes \tilde{f}_{i,l}(b') & \text{if } \phi_i(b) = 0. \end{cases}$

Proposition 3.5. $\mathcal{B} \otimes \mathcal{B}'$ *is a crystal.*

Proof. Note that the result is already known if $I^{\text{im}} = \varnothing$, hence we just have to check the axioms of 3.2 that concern imaginary vertices. Axioms (1), (2), (3), (4) and (7) are clearly satisfied. For the axiom (6), consider $i \in I^{\text{im}}$, $l > 0$ and b, b' such that $\tilde{f}_{i,l}(b \otimes b') \neq 0$. In particular, if $\phi_i(b) = +\infty$, $\tilde{f}_{i,l}(b) \neq 0$. Then :

$$\langle e_i, \text{wt}(\tilde{f}_{i,l}(b)) \rangle = \langle e_i, \text{wt}(b) \rangle - l\langle e_i, \alpha_i \rangle \geq \langle e_i, \text{wt}(b) \rangle > 0$$

since $\langle e_i, \alpha_i \rangle \leq 0$ for every $i \in I^{\text{im}}$, hence $\phi_i(\tilde{f}_{i,l}(b)) = +\infty$. By definition of the tensor product, we get $\epsilon_i(\tilde{f}_{i,l}(b) \otimes b') = \epsilon_i(\tilde{f}_{i,l}(b))$. But $\epsilon_i(b \otimes b') = \epsilon_i(b)$ in this case, hence the axiom (6) is satisfied if $\phi_i(b) = +\infty$. Otherwise $\phi_i(b) = 0$, and, by definition of the tensor product :

$$\epsilon_i(\tilde{f}_{i,l}(b \otimes b')) = \epsilon_i(b \otimes \tilde{f}_{i,l}(b')) = \epsilon_i(\tilde{f}_{i,l}(b')).$$

Since $\epsilon_i(b \otimes b') = \epsilon_i(b')$, the axiom (6) is still satisfied. The fact that the axiom (5) is satisfied can be proved in an analogous way. □

Notations 3.6. From 1.14, we have the following bijections :

$$\operatorname{Irr} \Lambda(\alpha)_{i,l} \xrightarrow[\sim]{\mathfrak{k}_{i,l}} \operatorname{Irr} \Lambda(\alpha - l e_i)_{i,0} \times \mathcal{C}_{i,l}$$

where $\alpha \in P$, $i \in I$, $l > 0$. Set, for $\mathsf{c} \in \mathcal{C}_{i,l}$:

$$\begin{aligned} \operatorname{Irr} \Lambda_{i,l} &= \bigsqcup_{\alpha \in P} \operatorname{Irr} \Lambda(\alpha)_{i,l} \\ \operatorname{Irr} \Lambda(\alpha)_{i,\mathsf{c}} &= \mathfrak{k}_{i,l}^{-1}(\operatorname{Irr} \Lambda(\alpha - l e_i)_{i,0} \times \{\mathsf{c}\}) \\ \operatorname{Irr} \Lambda_{i,\mathsf{c}} &= \bigsqcup_{\alpha \in P} \operatorname{Irr} \Lambda(\alpha)_{i,\mathsf{c}} \\ \operatorname{Irr} \Lambda &= \bigsqcup_{\alpha \in P} \operatorname{Irr} \Lambda(\alpha) \end{aligned}$$

and denote by $\tilde{e}_{i,\mathsf{c}}$ and $\tilde{f}_{i,\mathsf{c}}$ the inverse bijections :

$$\tilde{e}_{i,\mathsf{c}} : \operatorname{Irr} \Lambda_{i,\mathsf{c}} \rightleftarrows \operatorname{Irr} \Lambda_{i,0} : \tilde{f}_{i,\mathsf{c}}$$

induced by $\mathfrak{k}_{i,l}$. Then, for every $l > 0$, we define :

$$\begin{aligned} \tilde{e}_{i,l} &= \bigsqcup_{\mathsf{c} \in \mathfrak{C}_i} \delta_{\mathsf{c}_1,l} \tilde{f}_{i,\mathsf{c} \setminus \mathsf{c}_1} \tilde{e}_{i,\mathsf{c}} : \operatorname{Irr} \Lambda \to \operatorname{Irr} \Lambda \sqcup \{0\} \\ \tilde{f}_{i,l} &= \tilde{f}_{i,(l)} \sqcup \Big(\bigsqcup_{\mathsf{c} \in \mathcal{C}_i} \tilde{f}_{i,(l,\mathsf{c})} \tilde{e}_{i,\mathsf{c}} \Big) : \operatorname{Irr} \Lambda \to \operatorname{Irr} \Lambda \sqcup \{0\} \end{aligned}$$

where $\tilde{f}_{i,(l,\lambda)} = 0$ if $\omega_i = 1$ and $l > \lambda_1$.

It is obvious from the definitions that we have :

Proposition 3.7. *The set* $\operatorname{Irr} \Lambda$ *is a crystal with respect to* $\mathrm{wt} : b \in \operatorname{Irr} \Lambda(\alpha) \mapsto -C\alpha$*,* ϵ_i *the composition of* $\sqcup_{l>0} \mathfrak{k}_{i,l}$ *and the second projection, and* $\tilde{e}_{i,l}, \tilde{f}_{i,l}$ *the maps defined above.*

Notations 3.8. From 2.20, we have the following bijections :

$$\operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,l} \xrightarrow[\sim]{\mathfrak{l}_{i,l}} \operatorname{Irr} \mathfrak{L}(\mathsf{v} - l e_i, \mathsf{w})_{i,0} \times \mathcal{C}_{i,l}$$

where $\mathsf{v}, \mathsf{w} \in P$, $i \in I$, $l > 0$. Set, for $\mathsf{w} \in P$ and $\mathsf{c} \in \mathcal{C}_{i,l}$:

$$\begin{aligned} \operatorname{Irr} \mathfrak{L}(\mathsf{w})_{i,l} &= \bigsqcup_{\mathsf{v} \in P} \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,l} \\ \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,\mathsf{c}} &= \mathfrak{k}_{i,l}^{-1}(\operatorname{Irr} \mathfrak{L}(\mathsf{v} - l e_i, \mathsf{w})_{i,0} \times \{\mathsf{c}\}) \\ \operatorname{Irr} \mathfrak{L}(\mathsf{w})_{i,\mathsf{c}} &= \bigsqcup_{\mathsf{v} \in P} \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w})_{i,\mathsf{c}} \\ \operatorname{Irr} \mathfrak{L}(\mathsf{w}) &= \bigsqcup_{\mathsf{v} \in P} \operatorname{Irr} \mathfrak{L}(\mathsf{v}, \mathsf{w}) \end{aligned}$$

and denote by $\tilde{e}_{i,\mathsf{c}}$ and $\tilde{f}_{i,\mathsf{c}}$ the inverse bijections :

$$\tilde{e}_{i,\mathsf{c}} : \operatorname{Irr}\mathfrak{L}(\mathsf{w})_{i,\mathsf{c}} \rightleftarrows \operatorname{Irr}\mathfrak{L}(\mathsf{w})_{i,0} : \tilde{f}_{i,\mathsf{c}}$$

induced by $\mathfrak{l}_{i,l}$. Then, for every $l > 0$, we define :

$$\tilde{e}_{i,l} = \bigsqcup_{\mathsf{c}\in\mathcal{C}_i} \delta_{\mathsf{c}_1,l}\tilde{f}_{i,\mathsf{c}\setminus\mathsf{c}_1}\tilde{e}_{i,\mathsf{c}} : \operatorname{Irr}\mathfrak{L}(\mathsf{w}) \to \operatorname{Irr}\mathfrak{L}(\mathsf{w}) \sqcup \{0\}$$
$$\tilde{f}_{i,l} = \tilde{f}_{i,(l)} \sqcup \Big(\bigsqcup_{\mathsf{c}\in\mathcal{C}_i} \tilde{f}_{i,(l,\mathsf{c})}\tilde{e}_{i,\mathsf{c}} \Big) : \operatorname{Irr}\mathfrak{L}(\mathsf{w}) \to \operatorname{Irr}\mathfrak{L}(\mathsf{w}) \sqcup \{0\}$$

where $\tilde{f}_{i,(l,\lambda)} = 0$ if $\omega_i = 1$ and $l > \lambda_1$.

The following result is straightforward :

Proposition 3.9. *The set* Irr $\mathfrak{L}(\mathsf{w})$ *is a crystal with respect to* wt : $b \in \operatorname{Irr}\mathfrak{L}(\mathsf{v},\mathsf{w}) \mapsto \mathsf{w} - C\mathsf{v}$, ϵ_i *the composition of* $\sqcup_{l>0}\mathfrak{l}_{i,l}$ *and the second projection, and* $\tilde{e}_{i,l}$, $\tilde{f}_{i,l}$ *the maps defined above.*

Remark 3.10. Thanks to 2.19 and the classical case, we have, for every $i \in I$:

$$\phi_i(b) = \max\{|\mathsf{c}| \in \mathbb{N} \mid \tilde{f}_{i,\mathsf{c}}(b) \neq 0\}$$

where $\tilde{f}_{i,\mathsf{c}} = \tilde{f}_{i,\mathsf{c}_1} \dots \tilde{f}_{i,\mathsf{c}_r}$ $(= \tilde{f}_i^{|\mathsf{c}|}$ if i real).

In an analogous way, one can equip Irr $\widetilde{\mathfrak{Z}}$ with a structure of crystal, thanks to 2.27, and get :

Theorem 3.11. *The crystal structure on* Irr $\widetilde{\mathfrak{Z}}$ *coincides with that of the tensor product* Irr $\mathfrak{L}(\mathsf{w}) \otimes$ Irr $\mathfrak{L}(\mathsf{w}')$.

Proof. This is essentially 2.32. Note that for $b \in \operatorname{Irr}\mathfrak{L}(\mathsf{v},\mathsf{w})$, it is impossible to have $\mathsf{v}_i > 0$ and $\mathsf{w}_i + \sum_{h\in H_i} \mathsf{v}_{t(h)} = 0$, hence :

$$\mathsf{w}_i + \sum_{h\in H_i} \mathsf{v}_{t(h)} > 0 \Leftrightarrow \langle e_i, \mathsf{w} - C\mathsf{v}\rangle > 0.$$

□

Deuxième partie

Quivers with loops and perverse sheaves

Sommaire

Introduction

Lusztig defined in [Lus91] a *canonical basis* of the quantum group attached to any quiver without loop. This definition was possible thanks to an isomorphism between this quantum group and the Grothendieck group of a category of perverse sheaves, generated by the so-called *Lusztig sheaves*. Lusztig endowed this Grothendieck group with a structure of Hopf algebra, by means of restriction and induction functors. These functors made it possible for him to perform induction proofs via a nice stratification of his category. This construction yielded a combinatorial structure on the canonical basis which would later be recognized as a *Kashiwara crystal*.

There are more and more evidences of the relevance of the study of quivers with loops. A particular class of such quivers are the comet-shaped quivers, which have recently been used by Hausel, Letellier and Rodriguez-Villegas in their study of the topology of character varieties, where the number of loops at the central vertex is the genus of the considered curve (see [HRV08] and [HLRV13]). We can also see quivers with loops appearing in a work of Nakajima relating quiver varieties with branching (see [Nak09]), as in the work of Okounkov and Maulik about quantum cohomology (see [MO12]).

Kang and Schiffmann generalized Lusztig constructions in the framework of generalized Kac-Moody algebra in [KS06], using quivers with loops. In this case, one has to impose a somewhat unnatural restriction on the definition of a category of perverse sheaves, considering only those attached to complete flags on imaginary vertices.

In this article we consider the general definition of Lustig sheaves for arbitrary quivers, possibly carrying loops. We therefore follow the definition given in [Lus93], and use the results obtained in this article for quivers with one vertex and multiple loops. Note that the category hence considered is bigger than the one considered in [KS06], as one may already see in the case of the Jordan quiver. We prove a conjecture raised by Lusztig in [Lus93], asking if the more "simple" Lusztig perverse sheaves are enough to span the whole Grothendieck group considered. A partial proof was given in [LL09]. Our proof is also based on induction, still with the help of restriction and induction functors, but with non trivial first steps, consisting in the study of quivers with one vertex but possible loops. We also need to consider regularity conditions on the support of our perverse sheaves to perform efficient restrictions at imaginary vertices. From our proof emerges a new combinatorial structure on our generalized canonical basis, which is more general than the usual crystals, in that there are now more operators associated to a vertex with loops, as in [Boz13a] (see 1.13).

In a second part, we construct and study a Hopf algebra which generalizes the usual quantum groups. The geometric study previously made leads to a natural definition, which includes countably infinite sets of generators at imaginary roots, with higher order Serre relations and commutativity conditions imposed by the Jordan quiver case. We prove that the positive part of this algebra is isomorphic to our Grothendieck group, thanks to the study of a nondegenerate Hopf pairing.

In a final section, we try to build a bridge with the Lagrangian varieties studied in [Boz13a], using our new Hopf algebra, as the classical case suggests (see [Lus91]).

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1 Quiver Varieties

1.1 Preliminaries

Let Q be a quiver, with vertex set I and oriented edge set $\Omega = \{h : s(h) \to t(h)\}$. We will denote by $\Omega(i)$ the set of loops at i, and call i *imaginary* if $\omega_i = |\Omega(i)| \geq 1$, *real* otherwise.

1.1.1 Lusztig perverse sheaves

For every $\alpha = \sum_{i \in I} \alpha_i i \in \mathbb{N}I$, we fix an I-graded vector space V_α of graded dimension α. For every I-graded vector space X, we set :

$$E_X = \bigoplus_{h \in \Omega} \mathrm{Hom}(X_{s(h)}, X_{t(h)}),$$

and $E_\alpha = E_{V_\alpha}$. We also denote by G_α the group $\prod_{i \in I} GL(V_{\alpha_i i})$, naturally acting on E_α. Take $m > 0$ and two sequences $\mathbf{i} = (i_1, \ldots, i_m)$ and $\mathbf{a} = (a_1, \ldots, a_m)$ of I and $\mathbb{N}_{>0}$. We write $(\mathbf{i}, \mathbf{a}) \vdash \alpha$ if $\sum_{1 \leq k \leq m} a_k i_k = \alpha$. We set :

$$\mathcal{F}_{\mathbf{i},\mathbf{a}} = \left\{ \mathsf{W} = (\{0\} = \mathsf{W}_0 \subset \ldots \subset \mathsf{W}_m = V_\alpha) \;\middle|\; \forall k, \dim \frac{\mathsf{W}_k}{\mathsf{W}_{k-1}} = a_k i_k \right\}$$

$$\widetilde{E}_{\mathbf{i},\mathbf{a}} = \{(x, \mathsf{W}) \mid x_h(\mathsf{W}) \subseteq \mathsf{W}\} \subseteq E_\alpha \times \mathcal{F}_{\mathbf{i},\mathbf{a}}$$

so that we get a proper morphism $\pi_{\mathbf{i},\mathbf{a}} : \widetilde{E}_{\mathbf{i},\mathbf{a}} \to E_\alpha$ induced by the first projection.

Following [Lus10], we will denote by $\mathcal{M}_G(X)$ the category of G-equivariant perverse sheaves on an algebraic variety X equipped with an action of an algebraic connected group G.

Thanks to the decomposition theorem of Beilinson, Bernstein and Deligne (see [BBD82]), the complex $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ is semisimple. Denote by $\mathcal{P}_\alpha \subseteq \mathcal{M}_{G_\alpha}(E_\alpha)$ the additive category consisting of sums of G_α-equivariant simple perverse sheaves appearing (possibly with a shift) in $\pi_{\mathbf{i},\mathbf{a}!}\mathbf{1}$ for some $(\mathbf{i}, \mathbf{a}) \vdash \alpha$. Here $\mathbf{1}$ stands for the constant perverse sheaf on $\widetilde{E}_{\mathbf{i},\mathbf{a}}$.

Denote by $\mathcal{Q}_\alpha$ the category of complexes isomorphic to sums of shifts of sheaves of $\mathcal{P}_\alpha$.

Let $\mathcal{K}_\alpha$ be the Grothendieck group of $\mathcal{Q}_\alpha$, seen as a $\mathbb{Z}[v^{\pm 1}]$-module by setting $v^{\pm 1}[\mathbf{P}] = [\mathbf{P}[\pm 1]]$, $[\mathbf{P}]$ denoting the isoclass of a perverse sheaf $\mathbf{P}$. We will finally denote by $\mathcal{B}_\alpha$ the finite set of isoclasses of simple perverse sheaves in $\mathcal{P}_\alpha$, and we set $\mathcal{B} = \sqcup_\alpha \mathcal{B}_\alpha$.

1.1.2 Restriction and Induction functors

For every I-graded subspace $W \subseteq V_\alpha$ of dimension β and codimension γ, equipped with two I-graded isomorphisms $p : W \xrightarrow{\sim} V_\beta$ and $q : V_\alpha / W \xrightarrow{\sim} V_\gamma$, we have the following diagram :

$$E_\beta \times E_\gamma \xleftarrow{\kappa} E_\alpha(W) \xrightarrow{\iota} E_\alpha$$

where $E_\alpha(W) = \{x \in E_\alpha \mid x(W) \subseteq W\}$, $\kappa : x \mapsto (p_*(x_W), q_*(x_{V_\alpha/W}))$ and ι is the inclusion. Note that κ is a vector bundle.

We will also consider :

$$E_\beta \times E_\gamma \xleftarrow{p_1} E^\dagger_{\beta,\gamma} \xrightarrow{p_2} E_{\beta,\gamma} \xrightarrow{p_3} E_\alpha$$

where :

$$E^\dagger_{\beta,\gamma} = \left\{ (x, W, r, \bar{r}) \;\middle|\; \begin{array}{l} x \in E_\alpha \\ W \subseteq V_\alpha \text{ is } I\text{-graded and } x\text{-stable} \\ r : W \xrightarrow{\sim} V_\beta \\ \bar{r} : V_\alpha/W \xrightarrow{\sim} V_\gamma \end{array} \right\}$$
$$E_{\beta,\gamma} = \left\{ (x, W) \;\middle|\; \begin{array}{l} x \in E_\alpha \\ W \subseteq V_\alpha \text{ is } I\text{-graded and } x\text{-stable} \end{array} \right\}.$$

These diagrams induce (*cf.* [Lus10, §9.2]) :

$$\widetilde{\mathrm{Res}}_{\beta,\gamma} = \kappa_! \iota^* : \mathcal{Q}_\alpha \to \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta$$
$$\widetilde{\mathrm{Ind}}_{\beta,\gamma} = p_{3!} p_{2\flat} p_1^* : \mathcal{Q}_\gamma \boxtimes \mathcal{Q}_\beta \to \mathcal{Q}_\alpha$$

and :

$$\mathrm{Res}_{\beta,\gamma} = \widetilde{\mathrm{Res}}^\alpha_{\beta,\gamma}[d_1 - d_2 - 2\langle \beta, \gamma\rangle]$$
$$\mathrm{Ind}_{\beta,\gamma} = \widetilde{\mathrm{Ind}}^\alpha_{\beta,\gamma}[d_1 - d_2]$$

where d_1 and d_2 denote the dimensions of the fibers of p_1 and p_2, and $\langle \beta, \gamma\rangle = \sum_{i\in I} \beta_i \gamma_i$. These functors endow $\mathcal{K} = \oplus_\alpha \mathcal{K}_\alpha$ with a Hopf algebra structure (see [Lus91, 10]). Setting $(\gamma, \beta) = \sum_{h\in\Omega} \gamma_{s(h)} \beta_{t(h)}$, observe that :

$$d_1 - d_2 = (\gamma, \beta) + \langle \beta, \gamma\rangle$$
$$d_1 - d_2 - 2\langle \beta, \gamma\rangle = (\gamma, \beta) - \langle \beta, \gamma\rangle.$$

1.1.3 The geometric pairing

Let α be a dimension vector, and $\mathbf{P}$, $\mathbf{Q}$ be two G_α-equivariant semisimple complexes on E_α. Following [GL93], consider an integer $m \geq 1$ and a smooth irreducible algebraic variety Γ with a free action of G_α such that $H^j(\Gamma, \mathbb{C}) = 0$ if $1 \leq j \leq m$. The diagonal action of G_α on $\Gamma \times E_\alpha$ allows us to define ${}_\Gamma E_\alpha = G\backslash(\Gamma \times E_\alpha)$. Consider the following diagram, where the maps are the obvious ones :

$$E_\alpha \xleftarrow{s} \Gamma \times E_\alpha \xrightarrow{t} {}_\Gamma E_\alpha.$$

We can define two semisimple complexes ${}_\Gamma\mathbf{P}$ and ${}_\Gamma\mathbf{Q}$ on ${}_\Gamma E_\alpha$ by setting :

$$s^*\mathbf{P} = t^*{}_\Gamma\mathbf{P}$$
$$s^*\mathbf{Q} = t^*{}_\Gamma\mathbf{Q}.$$

By definition, we have :

$$H^j_{G_\alpha}(\mathbf{P}\otimes\mathbf{Q}, E_\alpha) = H^{2\dim(G\backslash\Gamma)-j}({}_\Gamma\mathbf{P}\otimes{}_\Gamma\mathbf{Q}, {}_\Gamma E_\alpha)$$

which is independant of Γ and $m \gg 0$. Define the geometric pairing by :

$$\{\mathbf{P},\mathbf{Q}\} = \sum_j \dim H^j_{G_\alpha}(\mathbf{P}\otimes\mathbf{Q}, E_\alpha)v^{-j}.$$

The following is true :

Proposition 1.1. *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are simple. We have :*

$$\{\mathbf{P},\mathbf{Q}\} \in 1 + v^{-1}\mathbb{N}[\![v^{-1}]\!]$$

if $\mathbf{Q}$ *is isomorphic to the Verdier dual of* $\mathbf{P}$*,*

$$\{\mathbf{P},\mathbf{Q}\} \in v^{-1}\mathbb{N}[\![v^{-1}]\!]$$

otherwise.

Recall that the Verdier dual keeps fixed any simple perverse sheaf in $\mathcal{P}_\alpha$.

1.2 Study of an imaginary sink

Let i be an imaginary sink, and $(\mathbf{i},\mathbf{a}) \vdash \alpha$. Take $\mathbf{a}_i = (a_{k_1},\ldots,a_{k_r})$ where $k_j < k_{j+1}$ and $\{k_j\}_{1\leq j\leq r} = \{k \mid i_k = i\}$. For $x \in E_\alpha$, we set $x^{(i)} = (x_h)_{h\in\Omega(i)}$ and $x^\diamond = (x_h)_{h\notin\Omega(i)}$. Then, we define :

$$\widetilde{E}^{(i)}_{\mathbf{i},\mathbf{a}} = \{(x,\mathsf{W}^{(i)}) \mid x^{(i)}(\mathsf{W}^{(i)}) \subseteq \mathsf{W}^{(i)}\} \subseteq E_\alpha \times \mathcal{F}^{(i)}_{\mathbf{a}_i}$$
$$E^\diamond_\alpha = \{x \in E_\alpha \mid x^{(i)} = 0\}$$

where $\mathcal{F}^{(i)}_{\mathbf{a}_i}$ denotes the variety of flags of $V_{\alpha_i i}$ of dimension $\mathbf{a}_i$. We have the following diagram :

$$\begin{tikzcd}
\widetilde{E}_{\mathbf{i},\mathbf{a}} \arrow[r, "\pi'_{\mathbf{i},\mathbf{a}}"] \arrow[rr, bend left, "\pi_{\mathbf{i},\mathbf{a}}"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\square"] & \widetilde{E}^{(i)}_{\mathbf{i},\mathbf{a}} \arrow[r, "\pi''_{\mathbf{i},\mathbf{a}}"] \arrow[d, "\mathrm{V}_{\mathbf{a}_i}"] & E_\alpha \\
\widetilde{E}^\diamond_{\mathbf{i},\mathbf{a}} \arrow[r, "\phi"'] & E^\diamond_\alpha \times \mathcal{F}^{(i)}_{\mathbf{a}_i} &
\end{tikzcd} \tag{1.2}$$

where $\widetilde{E}^\diamond_{\mathbf{i},\mathbf{a}} = \{(x,\mathsf{W}) \in \widetilde{E}_{\mathbf{i},\mathbf{a}} \mid x^{(i)} = 0\}$. Note that ψ and $\mathrm{V}_{\mathbf{a}_i}$ are vector bundles.

1.2.1 A notion of regularity

Put :

$$E_\alpha^{i,\mathrm{rss}} = \{x \in E_\alpha \mid x_h \text{ is regular semisimple if } h \in \Omega(i)\}.$$

For any constructible subsets $X \subseteq E_\alpha$, $Y \subseteq \widetilde{E}_{\mathbf{i},\mathbf{a}}$ and $Z \subseteq \widetilde{E}^{(i)}_{\mathbf{i},\mathbf{a}}$, we put :

$$\begin{aligned} X^{i,\mathrm{rss}} &= X \cap E_\alpha^{i,\mathrm{rss}} \\ Y^{i,\mathrm{rss}} &= Y \cap \pi_{\mathbf{i},\mathbf{a}}^{-1}(E_\alpha^{i,\mathrm{rss}}) \\ Z^{i,\mathrm{rss}} &= Z \cap {\pi''_{\mathbf{i},\mathbf{a}}}^{-1}(E_\alpha^{i,\mathrm{rss}}). \end{aligned}$$

We also write $\rho_\alpha : E_\alpha^{i,\mathrm{rss}} \hookrightarrow E_\alpha$ for the open inclusion.

Proposition 1.3. *Let* $\mathbf{P}$ *be any simple element of* $\mathcal{P}_\alpha$. *Then* $\mathbf{P} = \rho_{\alpha!*}\rho_\alpha^*\mathbf{P}$, *i.e. if* $\mathbf{P} = \mathrm{IC}(Y, \mathfrak{L})$ *for some smooth irreducible subvariety* $Y \subseteq E_\alpha$ *and some local system* $\mathfrak{L}$ *on* Y, *then* $Y^{i,\mathrm{rss}} \neq \varnothing$.

Proof. By definition, $\mathbf{P}$ appears as a simple summand of $\pi''_{\mathbf{i},\mathbf{a}!}\mathbf{Q}$ for some simple component $\mathbf{Q} \subseteq \pi'_{\mathbf{i},\mathbf{a}!}\mathbf{1}$. Since in 1.2 ψ is a vector bundle and the square is cartesian, $\mathbf{Q} \subseteq \mathsf{V}^*_{\mathbf{a}_i}\phi_!\mathbf{1}$, and thus $\mathbf{Q}$ is of the form $\mathrm{IC}(X, \mathfrak{K})$ where $X = \mathsf{V}^{-1}_{\mathbf{a}_i}(Y)$ for an irreducible smooth subvariety $Y \subseteq E_\alpha^\diamond \times \mathcal{F}^{(i)}_{\mathbf{a}_i}$, and $\mathfrak{K} = \mathsf{V}^*_{\mathbf{a}_i}\mathfrak{L}$ for an irreducible local system $\mathfrak{L}$ on Y.

In the lemma below, we call *quasismall* a map of algebraic varieties $\pi : X \to Y$ satisfying the following property : there exist stratifications $X = \sqcup_{j\in J} X_j$, $Y = \sqcup_{j \in J} Y_j$ over a finite set J containing an element 0 such that :

1. X_0 and Y_0 are dense ;
2. $\pi_{|X_j} : X_j \to Y_j$ is a locally trivial fibration of fiber F_j if $j \neq 0$;
3. $\pi_{|X_0} : X_0 \to Y_0$ is a finite morphism ;
4. $2 \dim F_j < \mathrm{codim}_Y Y_j$ if $j \neq 0$.

Lemma 1.4. *Let* S *be a smooth irreducible subvariety of* $E_\alpha^\diamond \times \mathcal{F}^{(i)}_{\mathbf{a}_i}$. *Put* $\widetilde{S} = \mathsf{V}^{-1}_{\mathbf{a}_i}(S)$ *and* $\bar{S} = \pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S})$. *Then the map* $\pi''_{\mathbf{i},\mathbf{a}|\widetilde{S}} : \widetilde{S} \to \bar{S}$ *is quasismall.*

Proof of the lemma. Put $\widetilde{S}^0 = \widetilde{S}^{i,\mathrm{rss}}$, which is a nonempty open dense subset of $\widetilde{S}$. Moreover, the restriction of $\pi''_{\mathbf{i},\mathbf{a}}$ to $\widetilde{S}^0$ is a finite morphism since a regular semisimple element x_h for $h \in \Omega(i)$ stabilizes only finitely many flags of subspaces of $V_{\alpha_i i}$. Put $\widetilde{T} = \widetilde{S} \setminus \widetilde{S}^0$. To prove that $\pi''_{\mathbf{i},\mathbf{a}|\widetilde{S}} : \widetilde{S} \to \bar{S}$ is quasismall, it now suffices to check that :

$$\dim(\widetilde{T} \times_{E_\alpha} \widetilde{T}) < \dim \widetilde{S}.$$

Let $z = (z_{h,k})$ be a $r \times r$-matrix of nonnegative integers such that $\sum_h z_{h,k} = a_k$, $\sum_k z_{h,k} = a_h$, and set :

$$(\widetilde{S} \times_{E_\alpha} \widetilde{S})_z = \left\{ (x, \mathsf{W}, \mathsf{W}') \;\middle|\; \forall h, k \ \dim \frac{\mathsf{W}_h \cap \mathsf{W}'_k}{\mathsf{W}_{h-1} \cap \mathsf{W}'_k + \mathsf{W}_h \cap \mathsf{W}'_{k-1}} = z_{h,k} \right\}.$$

This yields a finite stratification $\widetilde{S} \times_{E_\alpha} \widetilde{S} = \sqcup_z (\widetilde{S} \times_{E_\alpha} \widetilde{S})_z$. We use the same notations for $S \times_{E_\alpha^\diamond} S$ and $\widetilde{T} \times_{E_\alpha} \widetilde{T}$. The fibers of $\mathrm{V}_{\mathbf{a}_i|\widetilde{S}} : \widetilde{S} \to S$ being the same as those of $\widetilde{E}_{i,\mathbf{a}_i} \to \mathcal{F}_{\mathbf{a}_i}^{(i)}$, we have for any z as above :

$$\dim(\widetilde{S} \times_{E_\alpha} \widetilde{S})_z - \dim(S \times_{E_\alpha^\diamond} S)_z = \dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{E_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z - \dim(\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z \quad (1.5)$$

and :

$$\dim(\widetilde{T} \times_{E_\alpha} \widetilde{T})_z - \dim(S \times_{E_\alpha^\diamond} S)_z = \dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{U_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z - \dim(\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z$$

where $U_{\alpha_i i} = E_{\alpha_i i} \setminus E_{\alpha_i i}^{i,\mathrm{rss}}$. If $\omega_i = 1$, it is very well known that the map $\widetilde{E}_{i,\mathbf{a}_i} \to E_{\alpha_i i}$ is quasismall, with $E_{\alpha_i i}^{i,\mathrm{rss}}$ being the only relevant stratum. Indeed, it is true if $\mathbf{a}_i = (1^{\alpha_i})$, and we have the following commutative diagram :

$$\begin{tikzcd}
\widetilde{E}_{i,(1^{\alpha_i})} \arrow[rr, "f"] \arrow[dr, "g"'] & & E_{\alpha_i i} \\
& \widetilde{E}_{i,\mathbf{a}_i} \arrow[ur, "h"'] &
\end{tikzcd}$$

where g is projective, hence f quasismall implies h quasismall. It follows that :

$$\dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{U_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z < \dim \widetilde{E}_{i,\mathbf{a}_i}. \quad (1.6)$$

By [Lus93], this strict inequality is also true if $\omega_i \geq 2$. Indeed, the large inequality is true for any z if we replace $U_{\alpha_i i}$ by $E_{\alpha_i i}$, and, since $\dim U_{\alpha_i i} < \dim E_{\alpha_i i}$:

$$\dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{U_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z < \dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{E_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z \leq \dim \widetilde{E}_{i,\mathbf{a}_i},$$

hence 1.6 is still satisfied. But then :

$$\begin{aligned}
\dim \widetilde{S} - \dim(\widetilde{T} \times_{E_\alpha} \widetilde{T})_z &= \dim \widetilde{S} - \dim(S \times_{E_\alpha^\diamond} S)_z + \dim(S \times_{E_\alpha^\diamond} S)_z - \dim(\widetilde{S} \times_{E_\alpha} \widetilde{S})_z \\
&= \dim \widetilde{S} - \dim(S \times_{E_\alpha^\diamond} S)_z \\
&\qquad - \dim(\widetilde{E}_{i,\mathbf{a}_i} \times_{E_{\alpha_i i}} \widetilde{E}_{i,\mathbf{a}_i})_z + \dim(\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z \\
&\quad [\text{use } 1.5] \\
&> \dim \widetilde{S} - \dim(S \times_{E_\alpha^\diamond} S)_z - \dim \widetilde{E}_{i,\mathbf{a}_i} + \dim(\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z \\
&\quad [\text{use } 1.6] \\
&= \dim S - \dim(S \times_{E_\alpha^\diamond} S)_z - \dim \mathcal{F}_{\mathbf{a}_i}^{(i)} + \dim(\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z \\
&\quad [\text{use } 1.5 \text{ with } z \text{ diagonal}] \\
&= \operatorname{codim}_{((E_\alpha^\diamond \times \mathcal{F}_{\mathbf{a}_i}^{(i)}) \times_{E_\alpha^\diamond} (E_\alpha^\diamond \times \mathcal{F}_{\mathbf{a}_i}^{(i)}))_z} (S \times_{E_\alpha^\diamond} S)_z - \operatorname{codim}_{E_\alpha^\diamond \times \mathcal{F}_{\mathbf{a}_i}^{(i)}} S \\
&\geq 0,
\end{aligned}$$

the last inequality being true thanks to the following diagram :

$$\begin{tikzcd}
(S \times_{E_\alpha^\diamond} S)_z \arrow[r, hook] & \mathfrak{X} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & E_\alpha^\diamond \times (\mathcal{F}_{\mathbf{a}_i}^{(i)} \times \mathcal{F}_{\mathbf{a}_i}^{(i)})_z \arrow[d, "\mathrm{id} \times \mathrm{pr}_1"] \\
& S \arrow[r, hook] & E_\alpha^\diamond \times \mathcal{F}_{\mathbf{a}_i}^{(i)}
\end{tikzcd}$$

The lemma is proved. $\square$

End of proof of proposition 1.3. For any stratum $S \subseteq \overline{Y}$ for IC$(Y, \mathfrak{L})$, the subvariety $\widetilde{S} = \mathrm{V}_{\mathbf{a}_i}^{-1}(S)$ is a stratum for $\mathbf{Q}$. By 1.4, the restriction of $\pi''_{\mathbf{i},\mathbf{a}}$ to each of these strata is quasismall. By an argument identical to that in [KS07, 1], it follows that $\pi''_{\mathbf{i},\mathbf{a}!}\mathbf{Q}$ is a perverse sheaf, and that moreover any simple summand of $\pi''_{\mathbf{i},\mathbf{a}!}\mathbf{Q}$ is an intermediate extension to E_α of a simple direct summand of $\pi''_{\mathbf{i},\mathbf{a}!}(\mathrm{V}^*_{\mathbf{a}_i}(\mathfrak{L})_{|\widetilde{S}^0})$ for some irreducible local system $\mathfrak{L}$ on a stratum S. In particular, it is of the form IC$(R, \mathfrak{J})$ where R is an open subset of $\pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}^0)$ for some S, and $\mathfrak{J}$ is an irreducible local system on R. The proposition follows from the fact that, by construction, $\pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}^0) \subseteq E_\alpha^{i,\mathrm{rss}}$. $\square$

1.2.2 A notion of invariance

For any $x \in E_\alpha$, put $V_\alpha^\diamond = \oplus_{j \neq i} V_{\alpha_j j}$ and $\mathfrak{I}_i(x) = \mathbb{C}\langle x \rangle . V_\alpha^\diamond$, *i.e.* the smallest subspace of V_α stable by x and containing $V_\alpha^\diamond$.

Definition 1.7. Let us write $x \sim_i x'$ if the following holds :

1. $x^\diamond = x'^\diamond$;
2. $\mathfrak{I}_i(x) \subseteq \cap_{h \in \Omega(i)} \ker(x_h - x'_h)$;
3. $\sum_{h \in \Omega(i)} \mathrm{Im}(x_h - x'_h) \subseteq \mathfrak{I}_i(x)$.

Lemma 1.8. *$\sim_i$ is an equivalence relation.*

Proof.

- Reflexivity is obvious.
- Symmetry : if $x \sim_i x'$, then $\mathfrak{I}(x') = \mathfrak{I}(x)$ since $\mathbb{C}\langle x'^\diamond \rangle . V_\alpha^\diamond = \mathbb{C}\langle x^\diamond \rangle . V_\alpha^\diamond \subseteq \mathfrak{I}_i(x)$ and since $x^{(i)}_{|\mathfrak{I}_i(x)} = x'^{(i)}_{|\mathfrak{I}_i(x)}$. This implies $x' \sim_i x$.
- Transitivity : if $x \sim_i x'$ and $x' \sim_i x''$, we have $\mathfrak{I}_i(x) = \mathfrak{I}_i(x') = \mathfrak{I}_i(x'')$, $x^{(i)}_{|\mathfrak{I}_i(x)} = x'^{(i)}_{|\mathfrak{I}_i(x)} = x''^{(i)}_{|\mathfrak{I}_i(x)}$, and if $h \in \Omega(i)$:

$$\mathrm{Im}(x_h - x''_h) \subseteq \mathrm{Im}(x_h - x'_h) + \mathrm{Im}(x'_h - x''_h) \subseteq \mathfrak{I}_i(x).$$

  Hence $x \sim_i x''$.

$\square$

Observe that equivalence classes are affine spaces. If $x \in E_\alpha$, then the equivalence class of x is of dimension equal to $\omega_i \gamma(\alpha_i - \gamma)$ where $\omega_i = |\Omega(i)|$ and $\gamma i = \mathrm{codim}_{V_\alpha} \mathfrak{I}_i(x)$.

There is a stratification $E_\alpha = \sqcup_{\gamma \geq 0} E_{\alpha,i,\gamma}$ where :

$$E_{\alpha,i,\gamma} = \{x \in E_\alpha \mid \mathrm{codim}_{V_\alpha} \mathfrak{I}_i(x) = \gamma i\}.$$

Note that $E_{\alpha,i,\gamma}$ is a union of $\sim_i$-equivalence classes. This can be made more precise as follows. Fix $\gamma \leq \alpha_i$ and $W \subseteq V_\alpha$ an I-graded subspace of codimension γi. Let $E_{\alpha,i,\gamma}(W) = E_{\alpha,i,\gamma} \cap E_\alpha(W)$ be the closed subvariety of E_α of elements $x \in E_\alpha$ such that $\mathfrak{I}_i(x) = W$. Then, if $P = \mathrm{Stab}_{G_\alpha}(W)$,

$$E_{\alpha,i,\gamma} = G_\alpha \times_P E_{\alpha,i,\gamma}(W),$$

hence the inclusion $\iota_0 : E_{\alpha,i,\gamma}(W) \hookrightarrow E_{\alpha,i,\gamma}$ induces an equivalence of categories of perverse sheaves :

$$\iota_0^*[-d] : \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma}) \to \mathcal{M}_P(E_{\alpha,i,\gamma}(W))$$

where $d = \dim(G_\alpha/P)$. Observe also that $E_{\alpha,i,\gamma}(W)$ is itself a union of $\sim_i$-equivalence classes. Here ι_0 is a restriction of the inclusion ι introduced in 1.1, with γi in place of γ.

Now, as in 1.1, fix I-graded isomorphisms $W \simeq V_{\alpha-\gamma i}$ and $V_\alpha/W \simeq V_{\gamma i}$. We have a natural vector bundle map :

$$\kappa_0 : E_{\alpha,i,\gamma}(W) \to E_{\alpha-\gamma i,i,0} \times E_{\gamma i}$$

whose fibers are precisely the $\sim_i$-equivalence classes in $E_{\alpha,i,\gamma}(W)$. Again, κ_0 is a restriction of the vector bundle κ introduced in 1.1, with γi in place of γ. There is a fully faithful embedding :

$$\kappa_0^*[\omega_i d] : \mathcal{M}_{G_{\alpha-\gamma i}\times G_{\gamma i}}(E_{\alpha-\gamma i,i,0} \times E_{\gamma i}) \to \mathcal{M}_P(E_{\alpha,i,\gamma}(W)).$$

We say that a perverse sheaf $\mathbf{P} \in \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma})$ is *σ-invariant* (at i) if $\iota_0^*[-d](\mathbf{P})$ belongs to the essential image of $\kappa_0^*[\omega_i d]$.

Definition 1.9. Let $\mathcal{P}_{\alpha,i,\geq\gamma} \subseteq \mathcal{P}$ be the set of perverse sheaves supported on $E_{\alpha,i,\geq\gamma}$. The notation $\mathcal{P}_{\alpha,i,>\gamma}$ is defined likewise, and we set $\mathcal{P}_{\alpha,i,\gamma} = \mathcal{P}_{\alpha,i,\geq\gamma} \setminus \mathcal{P}_{\alpha,i,>\gamma}$. The terms $\mathcal{P}_{\alpha,i,\leq\gamma}$, $\mathcal{P}_{\alpha,i,<\gamma}$ are defined similarly.

We will need the following technical result :

Proposition 1.10. *Let $\mathbf{P}$ be any simple element of $\mathcal{P}_{\alpha,i,\gamma}$. Let $m : E_{\alpha,i,\gamma} \hookrightarrow E_{\alpha,i,\geq\gamma}$ be the open embedding. The perverse sheaf $m^*\mathbf{P} \in \mathcal{M}_{G_\alpha}(E_{\alpha,i,\gamma})$ is σ-invariant at i.*

Proof. The proof follows closely that of 1.3, whose notations we keep. In particular $\mathbf{P} = \mathrm{IC}(R, \mathfrak{J})$ where R is an open subset of $\pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}^0)$ for some G_α-invariant stratum $S \subseteq E_\alpha^\diamond \times \mathcal{F}_{\mathbf{a}_i}^{(i)}$. Moreover $\mathbf{P}$ appears in some complex :

$$\mathbf{R} = j_{*!}\Big(\pi''_{\mathbf{i},\mathbf{a}!}\big((\mathbf{V}_{\mathbf{a}_i}^*\mathfrak{L})_{|\widetilde{S}^0}\big)\Big)$$

where $j : \pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}^0) \hookrightarrow E_\alpha$ is the inclusion and where $\mathfrak{L}$ is a certain G_α-equivariant local system on S. It suffices to show that $\mathbf{R}$ is σ-equivariant.

Consider a stratification $S = \sqcup_k S(k)$ where :

$$S(k) = \{(x^\diamond, \mathsf{W}) \in S \mid \mathrm{Im}(x^\diamond) \cap V_{\alpha_i i} \subseteq \mathsf{W}_k \text{ but } \mathrm{Im}(x^\diamond) \cap V_{\alpha_i i} \not\subseteq \mathsf{W}_{k-1}\}.$$

Let k be maximal such that $S(k) \neq \varnothing$. Then $S(k)$ is open and dense in S. Denote by $\widetilde{S} = \sqcup_l \widetilde{S}(l)$ the induced stratification of $\widetilde{S}$. Then $\widetilde{S}(k)$ is also open and dense in $\widetilde{S}$. Finally, set :

$$\widetilde{S}(k)^\square = \{(x, \mathsf{W}) \in \widetilde{S}(k)^{i,\mathrm{rss}} \mid \mathfrak{I}_i(x) = \mathsf{W}_k\}.$$

It is easy to see that $\widetilde{S}(k)^\square$ is open and dense in $\widetilde{S}(k)$, hence in $\widetilde{S}$.

Put $\gamma = \sum_{l>k} \mathbf{a}_{il}$ so that $\gamma = \operatorname{codim}_{V_{\alpha_i i}} \mathsf{W}_k$ for any $\mathsf{W} \in \mathcal{F}^{(i)}_{\mathbf{a}_i}$. Let W an I-graded subspace of V_α of codimension γi with fixed identifications $W \simeq V_{\alpha-\gamma i}$ and $V_\alpha/W \simeq V_{\gamma i}$. Consider the following diagram :

$$\begin{tikzcd}
S(k) & \widetilde{S}(k)^\square \arrow[l, "\mathrm{V}_{\mathbf{a}_i}"'] \arrow[r, "\pi''_{\mathbf{i},\mathbf{a}}"] & E_{\alpha,i,\gamma} \\
S(k,W) \arrow[u, hook, "\bar\iota_0"] & \widetilde{S}(k,W)^\square \arrow[l, "\mathrm{V}_{\mathbf{a}_i}"'] \arrow[r, "\pi''_{\mathbf{i},\mathbf{a}}"] \arrow[u, hook, "\tilde\iota_0"] \arrow[d, "\tilde\kappa_0"] \arrow[ur, phantom, "\square"] \arrow[dr, phantom, "\square"] & E_{\alpha,i,\gamma}(W) \arrow[u, hook, "\iota_0"'] \arrow[d, "\kappa_0"] \\
 & \Xi \arrow[ul, "\exists\theta"] \arrow[r, "\pi''"] & E_{\alpha-\gamma i,i,0} \times E_{\gamma i}
\end{tikzcd} \tag{1.11}$$

where :

- $S(k,W) = \{(x^\diamond, \mathsf{W}) \mid \mathsf{W}_k = W\} \cap S(k) \subseteq S(k)$;
- $\widetilde{S}(k,W)^\square = \{(x, \mathsf{W}) \mid \mathsf{W}_k = W\} \cap \widetilde{S}(k)^\square \subseteq \widetilde{S}(k)^\square$;
- $\bar\iota_0$, $\tilde\iota_0$ and $\tilde\kappa_0$ stand for maps induced by ι_0 and κ_0 ;
- $\pi''_{\mathbf{i},\mathbf{a}}$ and $\mathrm{V}_{\mathbf{a}_i}$ (improperly) stand for maps induced by $\pi''_{\mathbf{i},\mathbf{a}}$ and $\mathrm{V}_{\mathbf{a}_i}$;
- $\Xi = \kappa(\widetilde{S}(k,W)^\square) \subseteq \widetilde{E}^{(i)}_{\mathbf{i}',\mathbf{a}'} \times \widetilde{E}^{(i)}_{\mathbf{i}'',\mathbf{a}''}$ where $(\mathbf{i}', \mathbf{a}') \vdash \alpha - \gamma i$ and $(\mathbf{i}'', \mathbf{a}'') \vdash \gamma i$ are naturally induced by $(\mathbf{i}, \mathbf{a})$ and k. Note the existence of an inclusion θ making commutative the triangle appearing in the diragram.
- π'' is the restriction of $\pi''_{\mathbf{i}',\mathbf{a}'} \times \pi''_{\mathbf{i}'',\mathbf{a}''}$ to Ξ.

Observe that the two rightmost squares are cartesian. This is obvious for the top square. For the bottom square, this follows from the fact that for $x \in E_{\alpha,i,\gamma}$, a flag $\mathsf{W} \in \mathcal{F}^{(i)}_{\mathbf{a}_i}$ satisfying $\mathsf{W}_k = \mathfrak{I}_i(x)$ is x-stable if and only if it is x'-stable for any $x' \sim_i x$.

Because $\widetilde{S}(k)^\square$ is open and dense in $\widetilde{S}^0$ and $\pi''_{\mathbf{i},\mathbf{a}|\widetilde{S}^0}$ is finite, we have :

$$\mathbf{R} = j'_{*!}\Big(\pi''_{\mathbf{i},\mathbf{a}!}\big((\mathrm{V}^*_{\mathbf{a}_i}\mathfrak{L})_{|\widetilde{S}(k)^\square}\big)\Big)$$

where $j' : \pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}(k)^\square) \hookrightarrow E_\alpha$ is the inclusion. Note that by construction $\mathbf{R}$ is a direct sum of objects in $\mathcal{P}_{\alpha,i,\gamma}$. We have :

$$m^*\mathbf{R} = j''_{*!}\Big(\pi''_{\mathbf{i},\mathbf{a}!}\big((\mathrm{V}^*_{\mathbf{a}_i}\mathfrak{L})_{|\widetilde{S}(k)^\square}\big)\Big)$$

where now j'' and m denote the inclusions defined by the following commmutative diagram :

$$\begin{tikzcd}
\pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}(k)^\square) \arrow[rr, hook, "j'"] \arrow[dr, hook, "j''"'] & & E_\alpha \\
 & E_{\alpha,i,\gamma} \arrow[ur, hook, "m"'] &
\end{tikzcd}$$

Furthermore, if $j''(W) : \pi''_{\mathbf{i},\mathbf{a}}(\widetilde{S}(k,W)^\square) \hookrightarrow E_{\alpha,i,\gamma}(W)$ denotes the inclusion induced by

j'',

$$\begin{aligned}
\iota_0^* m^* \mathbf{R} &= \iota_0^* j''_{*!} \pi''_{\mathbf{i},\mathbf{a}!}\big((\mathrm{V}^*_{\mathbf{a}_i}\mathfrak{L})_{|\widetilde{S}(k)^\square}\big) \\
&= j''(W)_{*!} \iota_0^* \pi''_{\mathbf{i},\mathbf{a}!}\big((\mathrm{V}^*_{\mathbf{a}_i}\mathfrak{L})_{|\widetilde{S}(k)^\square}\big) \\
&\qquad\qquad \text{[since } \iota_0^* \text{ is an equivalence of categories]} \\
&= j''(W)_{*!} \pi''_{\mathbf{i},\mathbf{a}!}\big((\mathrm{V}^*_{\mathbf{a}_i}\mathfrak{L})_{|\widetilde{S}(k,W)^\square}\big) \\
&\qquad\qquad \text{[the highest rightmost square in (1.11) being cartesian]} \\
&= j''(W)_{*!} \pi''_{\mathbf{i},\mathbf{a}!} \tilde{\kappa}_0^* \theta^*\big(\mathfrak{L}_{|S(k,W)}\big) \\
&\qquad\qquad \text{[the triangle being commutative in (1.11)]} \\
&= j''(W)_{*!} \kappa_0^* \pi''_! \theta^*\big(\mathfrak{L}_{|S(k,W)}\big) \\
&\qquad\qquad \text{[the lowest rightmost square in (1.11) being cartesian]} \\
&= \kappa_0^* \lambda_{*!} \pi''_! \theta^*\big(\mathfrak{L}_{|S(k,W)}\big)
\end{aligned}$$

where $\lambda : \pi''(\Xi) \hookrightarrow E_{\alpha-\gamma i,i,0} \times E_{\gamma i}$ is the inclusion (recall that κ_0 is a vector bundle). It follows that $m^*\mathbf{R}$ is σ-invariant as wanted. The proposition is proved. $\square$

1.3 A crystal type structure on $\mathcal{B}$

We keep the same notations. In particular, i is an imaginary sink and W is an I-graded subspace of V_α of codimension γi, with stabilizer $P \subseteq G_\alpha$. We also denote by U the unipotent radical of P.

Proposition 1.12. *Set $d = \dim(G_\alpha/P)$.*

(1) *Consider $A \in \mathcal{P}_{\alpha-\gamma i,i,0} \boxtimes \mathcal{P}_{\gamma i}$. For every n we have :*

$$\mathrm{supp}(H^n \mathrm{Ind}_{\alpha-\gamma i,\gamma i} A) \subseteq \overline{E_{\alpha,i,\gamma}}.$$

If $n \neq 0$, we have :

$$\mathrm{supp}(H^n \mathrm{Ind}_{\alpha-\gamma i,\gamma i} A) \cap E_{\alpha,i,\gamma} = \varnothing.$$

Otherwise, the sum of the simple components of $H^0\mathrm{Ind}_{\alpha-\gamma i,\gamma i}A$ belonging to $\mathcal{P}_{\alpha,i,\gamma}$ is nontrivial, and we denote it by $\xi(A)$.

(2) *Consider $B \in \mathcal{P}_{\alpha,i,\gamma}$. If $n \neq -2\omega_i d$, we have :*

$$\mathrm{supp}(H^n \mathrm{Res}_{\alpha-\gamma i,\gamma i} B) \cap E_{\alpha-\gamma i,i,0} \times E_{\gamma i} = \varnothing.$$

Otherwise, the sum of the simple components of $H^{-2\omega_i d}\mathrm{Res}_{\alpha-\gamma i,\gamma i}B$ belonging to $\mathcal{P}_{\alpha-\gamma i,i,0} \boxtimes \mathcal{P}_{\gamma i}$ is nontrivial, and we denote it by $\rho(B)$.

(3) *The functors ξ and ρ are equivalences of categories inverse to each other.*

Proof. We will use the following diagram :

$$\begin{tikzcd}
G_\alpha \times_P E_{\alpha,i,\gamma}(W) \arrow[r, "p_0", "\sim"'] \arrow[d, "m_0"'] & E_{\alpha,i,\gamma} \arrow[d, "m"'] & E_{\alpha,i,\gamma}(W) \arrow[l, "\iota_0"'] \arrow[r, "\kappa_0"] & E_{\alpha-\gamma i,i,0} \times E_{\gamma i} \arrow[d, "\mu"'] \\
G_\alpha \times_P E_\alpha(W) \arrow[r, "p=p_3"] & E_{\alpha,i,\geq\gamma} & E_\alpha(W) \arrow[l, "\iota"'] \arrow[r, "\kappa"] & E_{\alpha-\gamma i} \times E_{\gamma i}
\end{tikzcd}$$

To prove (1), we denote by $\tilde{A}$ the perverse sheaf $p_{2\flat}p_1^*A[(\omega_i+1)d]$. Therefore $\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A = p_!\tilde{A}[-(\omega_i+1)d]$, and thus the support of $\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A$ is included in the image of p, equal to $\overline{E_{\alpha,i,\gamma}}$. The following sheaf :

$$m^*\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A = m^*p_!\tilde{A}[-(\omega_i+1)d] = p_{0!}m_0^*\tilde{A}[-(\omega_i+1)d]$$

is perverse since m_0 is an open embedding, and since p_0 is an isomorphism. The support of $H^n\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A$ being included in $\overline{E_{\alpha,i,\gamma}}$ for all n, we get for $n \neq 0$:

$$m^*H^n\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A = H^n m^*\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A = 0$$

which proves (1) since $\widetilde{\mathrm{Ind}}_{\alpha-\gamma i,\gamma i}A[(\omega_i+1)d] = \mathrm{Ind}_{\alpha-\gamma i,\gamma i}A$.

To prove (2), we use the fact that m^*B is σ-equivariant, which implies that $\kappa_{0!}\iota_0^*m^*B[-(\omega_i+1)d]$ is perverse. But :

$$\begin{aligned}\kappa_{0!}\iota_0^*m^*B[-(\omega_i+1)d] &= \mu^*\kappa_!\iota^*B[-(\omega_i+1)d]\\ &= \mu^*\widetilde{\mathrm{Res}}_{\alpha-\gamma i,\gamma i}B[-(\omega_i+1)d],\end{aligned}$$

hence $\mu^*\mathrm{Res}_{\alpha-\gamma i,\gamma i}B[-2\omega_i d]$ is perverse. Since μ is an open embedding, we have, for $n \neq -2\omega_i d$:

$$\mu^*H^n\mathrm{Res}_{\alpha-\gamma i,\gamma i}B = H^n\mu^*\mathrm{Res}_{\alpha-\gamma i,\gamma i}B = 0$$

which ends the proof of (2).

We have the following diagram :

$$\begin{tikzcd}
E_{\alpha,i,\gamma}(W) \arrow[d, hook] & G_\alpha\times E_{\alpha,i,\gamma}(W) \arrow[l, "\mathrm{pr}_{2,0}"'] \arrow[r, "\pi_0^P"] \arrow[d, hook] & G_\alpha\times_P E_{\alpha,i,\gamma}(W) \arrow[d, hook] \\
E_\alpha(W) \arrow[d, "\kappa"'] & G_\alpha\times E_\alpha(W) \arrow[l, "\mathrm{pr}_2"'] \arrow[r, "\pi^P"] \arrow[d, "\pi^U"'] & G_\alpha\times_P E_\alpha(W) \\
E_{\alpha-\gamma i}\times E_{\gamma i} & G_\alpha\times_U E_\alpha(W) \arrow[l, "p_1"'] \arrow[ur, "p_2"'] &
\end{tikzcd}$$

where $\kappa\mathrm{pr}_2 = p_1\pi^U$ by definition of p_1, hence $\mathrm{pr}_2^*\kappa^* = \pi^{U*}p_1^*$, then $\pi_\flat^U\mathrm{pr}_2^*\kappa^* = p_1^*$, then $p_{2\flat}\pi_\flat^U\mathrm{pr}_2^*\kappa^* = p_{2\flat}p_1^*$ and thus :

$$\pi_\flat^P\mathrm{pr}_2^*\kappa^* = p_{2\flat}p_1^*$$

since $p_{2\flat}\pi_\flat^U = \pi_\flat^P$.

From the proof of (2) we have $\mu^*\rho(B) = \kappa_{0!}\iota_0^*m^*B[-(\omega_i+1)d]$, from which we get :

$$\begin{aligned}m_0^*\widetilde{\rho(B)} &= m_0^*p_{2\flat}p_1^*\rho(B)[(\omega_i+1)d]\\ &= m_0^*\pi_\flat^P\mathrm{pr}_2^*\kappa^*\rho(B)[(\omega_i+1)d]\\ &= \pi_{0\flat}^P\mathrm{pr}_{2,0}^*\kappa_0^*\mu^*\rho(B)[(\omega_i+1)d]\\ &= \pi_{0\flat}^P\mathrm{pr}_{2,0}^*\kappa_0^*\kappa_{0!}\iota_0^*m^*B\\ &= \pi_{0\flat}^P\mathrm{pr}_{2,0}^*\iota_0^*m^*B.\end{aligned}$$

But if we denote by $a, b : G_\alpha \times E_{\alpha,i,\gamma} \to E_{\alpha,i,\gamma}$ the action of G_α on $E_{\alpha,i,\gamma}$ and the second projection, we have :

$$\begin{aligned}
\pi_{0\flat}^P \mathrm{pr}_{2,0}^* \iota_0^* m^* B &= \pi_{0\flat}^P (\mathrm{id}_{G_\alpha} \times \iota_0)^* b^* m^* B \\
&= \pi_{0\flat}^P (\mathrm{id}_{G_\alpha} \times \iota_0)^* a^* m^* B \\
&\qquad\qquad [\text{by } G_\alpha\text{-equivariance of } B] \\
&= \pi_{0\flat}^P \pi_0^{P*} p_0^* m^* B \\
&\qquad\qquad [\text{by definition of } p_0] \\
&= p_0^* m^* B.
\end{aligned}$$

From the proof of (1), we also have $m^*\xi(A) = p_{0!} m_0^* \tilde{A}$, from which we get :

$$\begin{aligned}
\mu^* \rho(\xi(A)) &= \kappa_{0!} \iota_0^* m^* \xi(A)[-(\omega_i + 1)d] \\
&= \kappa_{0!} \iota_0^* p_{0!} m_0^* \tilde{A}[-(\omega_i + 1)d] \\
&= \kappa_{0!} \iota_0^* p_{0!} \pi_{0\flat}^P \mathrm{pr}_{2,0}^* \kappa_0^* \mu^* A
\end{aligned}$$

but we have seen earlier that for G_α-equivariant sheaves we have $\mathrm{pr}_{2,0}^* \iota_0^* = \pi_0^{P*} p_0^*$, hence $\iota_0^* p_{0!} = \mathrm{pr}_{2,0!} \pi_0^{P*}$, and thus :

$$\begin{aligned}
\mu^* \rho(\xi(A)) &= \kappa_{0!} \kappa_0^* \mu^* A \\
&= \mu^* A
\end{aligned}$$

but also :

$$\begin{aligned}
m^* \xi(\rho(B)) &= p_{0!} m_0^* \widetilde{\rho(B)} \\
&= p_{0!} p_0^* m^* B \\
&= m^* B.
\end{aligned}$$

We finally get (3). □

Proposition 1.13. *With the same hyoptheses and notations :*

1. *Let B be a simple object of $\mathcal{P}_{\alpha,i,\gamma}$. We have :*

$$\mathrm{Res}_{\alpha-\gamma i,\gamma i} B \simeq (A \boxtimes C) \oplus (\oplus_{j \in \mathbb{Z}} L_j[j])$$

   where A is a simple object of $\mathcal{P}_{\alpha-\gamma i,i,0}$, C a simple object of $\mathcal{P}_{\gamma i}$, and L_j is the tensor product of an element of $\mathcal{P}_{\alpha-\gamma i,i,>0}$ and an element of $\mathcal{P}_{\gamma i}$ for all j.

2. *Let (A, C) be a pair of simple objects of $\mathcal{P}_{\alpha-\gamma i,i,0} \times \mathcal{P}_{\gamma i}$. We have :*

$$\mathrm{Ind}_{\alpha-\gamma i,\gamma i}(A \boxtimes C) \simeq B \oplus (\oplus_{j \in \mathbb{Z}} L'_j[j])$$

   where B is a simple object of $\mathcal{P}_{\alpha,i,\gamma}$ and $L'_j \in \mathcal{P}_{\alpha,i,>\gamma}$ for all j.

3. *The maps $[B] \mapsto ([A],[C])$ and $([A],[C]) \mapsto [B]$ induced by (1) and (2) are inverse bijections between $\mathcal{B}_{\alpha,i,\gamma}$ and $\mathcal{B}_{\alpha-\gamma i,i,0} \times \mathcal{B}_{\gamma i}$.*

Proof. As in [Lus10, 10.3.2], the proof relies on 1.12, using the Fourier-Deligne transform (the result [Lus10, 10.3.1] remains true in our setting). □

We are now able to answer a question asked by Lusztig in [Lus93, 7]. We put $\mathbf{1}_{ai} = \pi_{i,a!}\mathbf{1}$:

Proposition 1.14. *The elements* $[\mathbf{1}_{ai}]$ *generate* $\mathcal{K}$ *(*$i \in I$*,* $a \in \mathbb{N}_{\geq 1}$*).*

Proof. We proceed by induction on α. Let B be a simple object of $\mathcal{P}_\alpha$. Using the Fourier-Deligne transform, we may assume that there is a sink i such that $B \in \mathcal{P}_{\alpha,i,\gamma}$ for some $\gamma > 0$ (see [Lus91, 7.2]). We then proceed by descending induction on γ. If i is real, we can conclude as in [Lus91, 7.3]. If i is imaginary, the second part of 1.13 together with the one vertex quiver case enable us to conclude. Indeed, the case of the Jordan quiver is well known (see *e.g.* [Sch09a]), and the case of the quiver with one vertex and multiple loops is treated in [Lus93]. □

2 A generalized quantum group

2.1 Generators

Let $(-,-)$ denote the symmetric Euler form on $\mathbb{Z}I$: (i,j) is equal to the opposite of the number of edges of Ω between i and j for $i \neq j \in I$, and $(i,i) = 2 - 2\omega_i$. We will denote by I^{re} (resp. I^{im}) the set of real (resp. imaginary) vertices, and by $I^{\text{iso}} \subseteq I^{\text{im}}$ the set of *isotropic* vertices : vertices i such that $(i,i) = 0$, *i.e.* such that $\omega_i = 1$. We also set $I_\infty = (I^{\text{re}} \times \{1\}) \cup (I^{\text{im}} \times \mathbb{N}_{\geq 1})$, and $(\iota, j) = l(i,j)$ if $\iota = (i,l) \in I_\infty$ and $j \in I$.

Definition 2.1. Let F denote the $\mathbb{Q}(v)$-algebra generated by $(E_\iota)_{\iota \in I_\infty}$, naturally $\mathbb{N}I$-graded by $\deg(E_{i,l}) = li$ for $(i,l) \in I_\infty$. We put $\mathsf{F}[A] = \{x \in \mathsf{F} \mid |x| \in A\}$ for any $A \subseteq \mathbb{N}I$, where, for convenience, we denote by $|x|$ the degree of an element x.

For $\alpha = \sum \alpha_i i \in \mathbb{Z}I$, we set :

▷ $\text{ht}(\alpha) = \sum \alpha_i$ its height ;

▷ $v_\alpha = \prod v_i^{\alpha_i}$ if $v_i = v^{(i,i)/2}$.

We endow $\mathsf{F} \otimes \mathsf{F}$ with the following multiplication :

$$(a \otimes b)(c \otimes d) = v^{(|b|,|c|)}(ac) \otimes (bd).$$

and equip F with a comultiplication δ defined by :

$$\delta(E_{i,l}) = \sum_{t+t'=l} v_i^{tt'} E_{i,t} \otimes E_{i,t'}$$

where $(i,l) \in I_\infty$.

Proposition 2.2. *For any family* $(\nu_\iota)_{\iota \in I_\infty}$*, we can endow* F *with a bilinear form* $\langle -,- \rangle$ *such that :*

▷ $\langle x,y \rangle = 0$ *if* $|x| \neq |y|$ *;*

▷ $\langle E_\iota, E_\iota \rangle = \nu_\iota$ *for all* $\iota \in I_\infty$ *;*

▷ $\langle ab, c \rangle = \langle a \otimes b, \delta(c) \rangle$ *for all* $a,b,c \in \mathsf{F}$.

Proof. Strictly analogous to [Lus10, Proposition 1.2.3] or [Rin96, 3]. □

Notations 2.3. Take $i \in I^{\text{im}}$ and c a composition (*i.e.* a tuple of positive integers) or a partition (*i.e.* a decreasing tuple of positive integers). We put $E_{i,\mathsf{c}} = \prod_j E_{i,\mathsf{c}_j}$, $\nu_{i,\mathsf{c}} = \prod_j \nu_{i,\mathsf{c}_j}$, and $|\mathsf{c}| = \sum \mathsf{c}_j$.

2.2 Relations

Proposition 2.4. *Consider $(\iota, j) \in I_\infty \times I^{re}$. The element :*

$$\sum_{t+t'=-(\iota,j)+1} (-1)^t E_j^{(t)} E_\iota E_j^{(t')} \tag{2.5}$$

belongs to the radical of $\langle -, - \rangle$.

Proof. Analogous to [Lus10, Proposition 1.4.3] or [Rin97]. □

Remark 2.6. Some higher order Serre relations are studied in [Lus10, Chapter 7], where some conditions are given to belong to the radical. However the proofs cannot be directly adapted to our setting.

The following definition is motivated by the previous proposition and our knowledge of the Jordan quiver case, which is related to the classical Hall algebra (see *e.g.* [Sch09b]). We know that the commutators $[E_{i,l}, E_{i,k}]$ lie in the radical if i is isotropic.

Definition 2.7. We denote by $\tilde{U}^+$ the quotient of F by the ideal spanned by the elements 2.5 and the commutators $[E_{i,l}, E_{i,k}]$ for every isotropic vertex i, so that $\langle -, - \rangle$ is still defined on $\tilde{U}^+$. We denote by U^+ the quotient of $\tilde{U}^+$ by the radical of $\langle -, - \rangle$.

Definition 2.8. Let $\hat{U}$ be the quotient of the algebra generated by $K_i^{\pm}$, E_ι, F_ι ($i \in I$ and $\iota \in I_\infty$) subject to the following relations :

$$\begin{aligned}
K_i K_j &= K_j K_i \\
K_i K_i^- &= 1 \\
K_j E_\iota &= v^{(j,\iota)} E_\iota K_j \\
K_j F_\iota &= v^{-(j,\iota)} F_\iota K_j \\
\sum_{t+t'=-(\iota,j)+1} (-1)^t E_j^{(t)} E_\iota E_j^{(t')} &= 0 \qquad (j \in I^{\mathrm{re}}) \\
\sum_{t+t'=-(\iota,j)+1} (-1)^t F_j^{(t)} F_\iota F_j^{(t')} &= 0 \qquad (j \in I^{\mathrm{re}}) \\
[E_{i,l}, E_{i,k}] &= 0 \qquad (i \in I^{\mathrm{iso}}) \\
[F_{i,l}, F_{i,k}] &= 0 \qquad (i \in I^{\mathrm{iso}}).
\end{aligned}$$

We extend the graduation by $|K_i| = 0$ and $|F_\iota| = -|E_\iota|$, and we set $K_\alpha = \prod_i K_i^{\alpha_i}$ for every $\alpha \in \mathbb{Z}I$.

We endow $\hat{U}$ with a comultiplication Δ defined by :

$$\begin{aligned}
\Delta(K_i) &= K_i \otimes K_i \\
\Delta(E_{i,l}) &= \sum_{t+t'=l} v_i^{tt'} E_{i,t} K_{t'i} \otimes E_{i,t'} \\
\Delta(F_{i,l}) &= \sum_{t+t'=l} v_i^{-tt'} F_{i,t} \otimes K_{-ti} F_{i,t'}.
\end{aligned}$$

We extend $\langle -,-\rangle$ to the subalgebra $\hat{U}^{\geq 0} \subseteq \hat{U}$ spanned by $(K_i^{\pm})_{i\in I}$ and $(E_\iota)_{\iota\in I_\infty}$ by setting $\langle xK_i, yK_j\rangle = \langle x,y\rangle v^{(i,j)}$ for $x,y \in \tilde{U}^+$.

We use the Drinfeld double process to define $\tilde{U}$ as the quotient of $\hat{U}$ by the relations :

$$\sum \langle a_{(1)}, b_{(2)}\rangle \omega(b_{(1)}) a_{(2)} = \sum \langle a_{(2)}, b_{(1)}\rangle a_{(1)} \omega(b_{(2)}) \tag{2.9}$$

for any $a,b \in \tilde{U}^{\geq 0}$, where ω is the unique involutive automorphism of $\hat{U}$ mapping E_ι to F_ι and K_i to K_{-i}, and where we use the Sweedler notation, for example $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

Setting $x^- = \omega(x)$ for $x \in \tilde{U}$, we define $\langle -,-\rangle$ on the subalgebra $\tilde{U}^- \subseteq \tilde{U}$ spanned by $(F_\iota)_{\iota\in I_\infty}$ by setting $\langle x,y\rangle = \langle x^-, y^-\rangle$ for any $x,y \in \tilde{U}^-$. We will denote by U^- (resp. U) the quotient of $\tilde{U}^-$ (resp. $\tilde{U}$) by the radical of $\langle -,-\rangle$ restricted to $\tilde{U}^-$(resp. restricted to $\tilde{U}^- \times \tilde{U}^+$).

Proposition 2.10. [Xia97] *We can define $S, S^{\mathrm{op}} : U \to U^{\mathrm{op}}$ (the* antipode *and the* skew antipode*) such that :*

$$\mathbf{m}(S\otimes \mathbf{1})\Delta = \mathbf{m}(\mathbf{1}\otimes S)\Delta = \epsilon\mathbf{1}$$
$$\mathbf{m}(S^{\mathrm{op}}\otimes \mathbf{1})\Delta^{\mathrm{op}} = \mathbf{m}(\mathbf{1}\otimes S^{\mathrm{op}})\Delta^{\mathrm{op}} = \epsilon\mathbf{1},$$

where $\mathbf{m}$ denotes the multiplication, ϵ denotes the counit, which is equal to 1 on U^0, and 0 on $U^- \times U^+$, and Δ^{op} denotes the composition of Δ and $\mathrm{op} : U\otimes U \to U \otimes U$, $x\otimes y \mapsto y \otimes x$. We also know that $S^{\mathrm{op}} = S^{-1}$.

2.3 The case of the quiver with one vertex and multiple loops

Lemma 2.11. *We have $\langle E_{i,|\mathsf{c}|}, E_{i,\mathsf{c}}\rangle = v_i^{\sum_{k<j} \mathsf{c}_k \mathsf{c}_j} \nu_{i,\mathsf{c}}$.*

Proof. By induction, using the definitions. □

Proposition 2.12. *Let $i \in I$ be a nonisotropic imaginary vertex. Assume that for every $l \geq 1$ we have :*

$$\langle E_{i,l}, E_{i,l}\rangle \in 1 + v^{-1}\mathbb{N}[\![v^{-1}]\!]. \tag{2.13}$$

Then, for any compositions c and c',

$$\langle E_{i,\mathsf{c}}, E_{i,\mathsf{c}'}\rangle \in \delta_{\mathsf{c},\mathsf{c}'} + v^{-1}\mathbb{N}[\![v^{-1}]\!].$$

Proof. For clarity, we forget the indices i in this proof. Notice that by definition of δ, of the multiplication on $\mathsf{F}\otimes\mathsf{F}$, and since $(i,i) < 0$, we already have :

$$\langle E_{\mathsf{c}}, E_{\mathsf{c}'}\rangle \in \mathbb{N}[\![v^{-1}]\!].$$

Hence, we can work modulo v^{-1}, and then, setting $\mathsf{c} = (\mathsf{c}_1, \dots, \mathsf{c}_r)$, $\mathsf{c}' = (\mathsf{c}'_1, \dots, \mathsf{c}'_s)$, $\tilde{\mathsf{c}} = (\mathsf{c}_2, \dots, \mathsf{c}_r)$ and $\tilde{\mathsf{c}}' = (\mathsf{c}'_2, \dots, \mathsf{c}'_s)$, we get :

$$\begin{aligned}\langle E_{\mathsf{c}}, E_{\mathsf{c}'}\rangle &= \left\langle E_{\mathsf{c}_1} \otimes E_{\tilde{\mathsf{c}}}, \prod_{1\leq j\leq s} \delta(E_{\mathsf{c}'_j})\right\rangle \\ &= \left\langle E_{\mathsf{c}_1} \otimes E_{\tilde{\mathsf{c}}}, \prod_{1\leq j\leq s} (E_{\mathsf{c}'_j} \otimes 1 + 1 \otimes E_{\mathsf{c}'_j})\right\rangle \bmod v^{-1} \\ &= \begin{cases} 0 \bmod v^{-1} & \text{if } \mathsf{c}'_1 \neq \mathsf{c}_1 \\ \langle E_{\tilde{\mathsf{c}}}, E_{\tilde{\mathsf{c}}'}\rangle \bmod v^{-1} & \text{otherwise} \end{cases}\end{aligned}$$

the second equality coming from the definition of δ, and from $(i,i) < 0$; the last equality coming from the definition of the multiplication on $\mathsf{F} \otimes \mathsf{F}$, from $(i,i) < 0$, from 2.11, and from the hypothesis of the proposition. We end the proof by induction. □

Corollary 2.14. *Under the assumption 2.13, the restriction of $\langle -,-\rangle$ to $\mathsf{F}[\mathbb{N}i]$ is nondegenerate.*

Remark 2.15. Under the same assumption, we know that if i is isotropic, the radical is exactly the ideal spanned by the commutators $[E_{i,l}, E_{i,k}]$. Hence in the sequel, we will assume 2.13 is satisfied for any imaginary vertex.

Notations 2.16. We denote by $\mathcal{C}_{i,l}$ the set of compositions c (resp. partitions) such that $|\mathsf{c}| = l$ if $(i,i) < 0$ (resp. $(i,i) = 0$).

2.4 Quasi $\mathcal{R}$-matrix

Proposition 2.17. *For any imaginary vertex i and any $l \geq 1$, there exists a unique element $a_{i,l} \in \mathsf{F}[li]$ such that, if we set $b_{i,l} = a_{i,l}^{-}$, we get :*

1. *$\langle E_{i,l} \mid l \geq 1\rangle = \langle a_{i,l} \mid l \geq 1\rangle$ and $\langle F_{i,l} \mid l \geq 1\rangle = \langle b_{i,l} \mid l \geq 1\rangle$ as algebras ;*
2. *$\langle a_{i,l}, z\rangle = \langle b_{i,l}, z^-\rangle = 0$ for any $z \in \langle E_{i,k} \mid k < l\rangle$;*
3. *$a_{i,l} - E_{i,l} \in \langle E_{i,k} \mid k < l\rangle$ and $b_{i,l} - F_{i,l} \in \langle F_{i,k} \mid k < l\rangle$;*
4. *$\bar{a}_{i,l} = a_{i,l}$ and $\bar{b}_{i,l} = b_{i,l}$;*
5. *$\Delta(a_{i,l}) = a_{i,l} \otimes 1 + K_{li} \otimes a_{i,l}$ and $\Delta(b_{i,l}) = b_{i,l} \otimes K_{-li} + 1 \otimes b_{i,l}$;*
6. *$S(a_{i,l}) = -K_{-li}a_{i,l}$ and $S(b_{i,l}) = -b_{i,l}K_{li}$.*

Proof. The properties 2 and 3 enable us to define $a_{i,l}$ uniquely, and imply the other ones. □

Notations 2.18. Consider $i \in I^{\mathrm{im}}$ and $\mathsf{c} \in \mathcal{C}_{i,l}$. We set $\tau_{i,l} = \langle a_{i,l}, a_{i,l}\rangle$, $a_{i,\mathsf{c}} = \prod_j a_{i,\mathsf{c}_j}$, and $\tau_{i,\mathsf{c}} = \prod_j \tau_{i,\mathsf{c}_j}$. Notice that $\{a_{i,\mathsf{c}} \mid \mathsf{c} \in \mathcal{C}_{i,l}\}$ is a basis of $\mathsf{F}[li]$.

Definition 2.19. We denote by $\delta_{i,\mathsf{c}}, \delta^{i,\mathsf{c}} : \mathsf{F} \to \mathsf{F}$ the linear maps defined by :

$$\delta(x) = \sum_{\mathsf{c}\in\mathcal{C}_{i,l}} \delta_{i,\mathsf{c}}(x) \otimes a_{i,\mathsf{c}} + \text{obd}$$

$$\delta(x) = \sum_{\mathsf{c}\in\mathcal{C}_{i,l}} a_{i,\mathsf{c}} \otimes \delta^{i,\mathsf{c}}(x) + \text{obd}$$

where "obd" stands for terms of bidegree not in $\mathbb{N}I \times \mathbb{N}i$ in the former equality, $\mathbb{N}i \times \mathbb{N}I$ in the latter one.

Proposition 2.20. *The maps $\delta_{i,\mathsf{c}}$ and $\delta^{i,\mathsf{c}}$ preserve the radical of $\langle -,-\rangle$.*

Proof. First consider the case where i is isotropic and x is a commutator $[E_{i,l}, E_{i,k}]$, then we have $\delta(x) = 0$, and thus $\delta_{i,\mathsf{c}}(x) = \delta^{i,\mathsf{c}}(x) = 0$. Thus, we can assume that $\langle -,-\rangle$ is nondegenerate on $\mathsf{F}[\mathbb{N}i]$. Consider x in the radical of $\langle -,-\rangle$. If $|\mathsf{c}| = l$, we have, for all $y \in \mathsf{F}$:

$$\begin{aligned}0 &= \langle x, ya_{i,\mathsf{c}}\rangle \\ &= \langle \delta(x), y \otimes a_{i,\mathsf{c}}\rangle \\ &= \sum_{|\mathsf{c}'|=l} \langle \delta_{i,\mathsf{c}'}(x) \otimes a_{i,\mathsf{c}'}, y \otimes a_{i,\mathsf{c}}\rangle \\ &= \sum_{|\mathsf{c}'|=l} \langle \delta_{i,\mathsf{c}'}(x), y\rangle\langle a_{i,\mathsf{c}'}, a_{i,\mathsf{c}}\rangle.\end{aligned}$$

The result comes from the nondegeneracy of the restriction of $\langle -,-\rangle$ to $\mathsf{F}[\mathbb{N}i]$. ☐

Lemma 2.21. *We have :*

1. $\langle a_{i,l}, a_{i,\mathsf{c}}\rangle = \delta_{(l),\mathsf{c}}\tau_{i,l}$ *;*
2. $\langle a_{i,l}y, z\rangle = \tau_{i,l}\langle y, \delta^{i,l}(z)\rangle$ *for any $y, z \in \mathsf{F}$;*
3. $\langle ya_{i,l}, z\rangle = \tau_{i,l}\langle y, \delta_{i,l}(z)\rangle$ *for any $y, z \in \mathsf{F}$.*

Proof. The first point is a direct consequence of the definition of the $a_{i,l}$, and the rest comes from it. ☐

Definition 2.22. Let $U\hat{\otimes}U$ be the completion of $U \otimes U$ with respect to the following sequence $(t \geq 1)$:

$$\mathcal{F}_t = \Big(U^+U^0 \sum_{|\alpha|\geq t} U^-[\alpha]\Big) \otimes U + U \otimes \Big(U^-U^0 \sum_{|\alpha|\geq t} U^+[\alpha]\Big).$$

Proposition 2.23. *For any $\alpha \in \mathbb{N}I$, let B_α be a basis of $U^+[\alpha] = \{x \in U^+, |x| = \alpha\}$, and $\{b^*|b \in B_\alpha\}$ the dual basis with respect to $\langle -,-\rangle$. Set :*

$$\Theta_\alpha = \sum_{b\in B_\alpha} b^- \otimes b^*.$$

Then, the element $\Theta = \sum \Theta_\alpha \in U\hat{\otimes}U$ satisfies :

$$\Delta(u)\Theta = \Theta\bar{\Delta}(u) \text{ for all } u \in U$$

where $\bar{\Delta}(u) = \overline{\Delta(\overline{u})}$ if $u \mapsto \overline{u}$ denotes the unique involutive $\mathbb{Q}$-morphism of U stabilizing E_ι and F_ι, and mapping K_i to K_{-i}, and v to v^{-1}.

Proof. It's enough to check the relation on generators. For those of real degree, the proof is identical to the one of [Lus10, Theorem 4.1.2]. Consider $i \in I^{\mathrm{im}}$ and $l \geq 1$. We have :

$$\Delta(a_{i,l})\Theta = \Theta\bar{\Delta}(a_{i,l}) \Leftrightarrow \sum_{b\in B}\{a_{i,l}b^- \otimes b^* + K_{li}b^- \otimes a_{i,l}b^* \\ - b^-a_{i,l} \otimes b^* - b^-K_{-li} \otimes b^*a_{i,l}\} = 0$$

$$\Leftrightarrow \forall z \in U^+, \sum_{b\in B}\{a_{i,l}b^-\langle b^*, z\rangle + K_{li}b^-\langle a_{i,l}b^*, z\rangle - b^-a_{i,l}\langle b^*, z\rangle - b^-K_{-li}\langle b^*a_{i,l}, z\rangle\} = 0$$

$$\Leftrightarrow \forall z \in U^+, \sum_{b\in B}\{a_{i,l}b^-\langle b^*, z\rangle + K_{li}b^-\tau_{i,l}\langle b^*, \delta^{i,l}(z)\rangle - b^-a_{i,l}\langle b^*, z\rangle - b^-K_{-li}\tau_{i,l}\langle b^*, \delta_{i,l}(z)\rangle\} = 0$$

$$\Leftrightarrow \forall z \in U^+,\ a_{i,l}z^- + \tau_{i,l}K_{li}\delta^{i,l}(z)^- = z^-a_{i,l} + \tau_{i,l}\delta_{i,l}(z)^-K_{-li}$$

which is the relation (2.9) with $a, b = a_{i,l}, z$. The equivalence before the last one comes from 2.21. The computations are the same for $U^{\leq 0}$:

$$\Delta(b_{i,l})\Theta = \Theta\bar{\Delta}(b_{i,l}) \Leftrightarrow \sum_{b\in B}\{b_{i,l}b^- \otimes K_{-li}b^* + b^- \otimes b_{i,l}b^* - b^-b_{i,l} \otimes b^*K_{li} - b^- \otimes b^*b_{i,l}\} = 0$$

$$\Leftrightarrow \forall z \in U^+, \sum_{b\in B}\{\langle a_{i,l}b, z\rangle K_{-li}b^* + \langle b, z\rangle b_{i,l}b^* - \langle ba_{i,l}, z\rangle b^*K_{li} - \langle b, z\rangle b^*b_{i,l}\} = 0$$

$$\Leftrightarrow \forall z \in U^+, \sum_{b\in B}\{\tau_{i,l}\langle b, \delta^{i,l}(z)\rangle K_{-li}b^* + \langle b, z\rangle b_{i,l}b^* - \tau_{i,l}\langle b, \delta_{i,l}(z)\rangle b^*K_{li} - \langle b, z\rangle b^*b_{i,l}\} = 0$$

$$\Leftrightarrow \forall z \in U^+,\ \tau_{i,l}K_{-li}\delta^{i,l}(z) + b_{i,l}z = \tau_{i,l}\delta_{i,l}(z)K_{li} + zb_{i,l}$$

which matches $(2.9)^-$ with $a, b = a_{i,l}, z$. □

Remark 2.24. As in [Lus10, 4.1.2], one can prove that Θ is the only element satisfying $\Theta_0 = 1 \otimes 1$ and $\Delta(u)\Theta = \Theta\bar{\Delta}(u)$ for all $u \in U$.

2.5 Casimir operator

Definition 2.25. We denote by $\mathcal{C}$ the category of U-modules satisfying :

1. $M = \oplus_{\alpha\in\mathbb{Z}I}M_\alpha$ where $M_\alpha = \{m \in M \mid \forall i,\ K_im = v^{(\alpha,i)}m\}$;
2. For any $m \in M$, there exists $p \geq 0$ such that $xm = 0$ as soon as $x \in \mathsf{F}[\alpha]$ and $\mathrm{ht}(\alpha) \geq p$.

Proposition 2.26. *Set* $\Omega_{\leq p} = \mathbf{m}(S \otimes \mathbf{1})(\sum_{\mathrm{ht}(\alpha)\leq p}\Theta_\alpha)$*, and* $M \in \mathcal{C}$*. Then, for every* $m \in M$*, the value of* $\Omega(m) = \Omega_{\leq p}(m)$ *does not depend on* p *for* p *large enough, and we have the following identities of operators on* M *:*

$$K_i\Omega = \Omega K_i$$
$$K_{-li}a_{i,l}\Omega = K_{li}\Omega a_{i,l}$$
$$b_{i,l}K_{li}\Omega K_{li} = \Omega b_{i,l}$$

for any $i \in I$ *and* $l \geq 1$.

Proof. The computations are strictly analogous to those in [Lus10, 6.1.1], thanks to the definition of $a_{i,l}$ and $b_{i,l}$ (see 2.17). $\square$

Definition 2.27. For any $\alpha \in \mathbb{Z}I$, we define a *Verma module* :

$$M(\alpha) = \frac{U}{\sum_{\iota \in I_\infty} UE_\iota + \sum_{i \in I} U(K_i - v^{(i,\alpha)})} \in \mathcal{C}.$$

Proposition 2.28. *Under the assumption 2.13, we have $\tilde{U}^- \simeq U^-$.*

Proof. The proof follows [Kac90], [Lus10] and more specifically [SVDB01, Proposition 2.4]. The maximal degrees of the primitive elements of the kernel of the map $\tilde{U}^- \to U^-$ are the same as those of the primitive elements of :

$$\ker \Big(\sum_{(i,l) \in I_\infty} \bullet \ b_{i,l} : \bigoplus_{(i,l) \in I_\infty} M(-li) \to M(0) \Big).$$

By maximality, if α is such a degree, we get $(\alpha, i) \geq 0$ for any vertex i. Indeed, [SVDB01, §2, properties 1.,2.,3.,4.] are still satisfied in our case, in particular the second one, thanks to the higher order Serre relations.

Let C denote the $\mathbb{Q}(v)$-linear map defined on $M = \oplus_{(i,l) \in I_\infty} M(-li)$ by :

$$Cm = v^{f(\alpha)} \Omega m \text{ if } m \in M_\alpha,$$

where $f(\alpha) = (\alpha, \alpha + 2\rho)$ and ρ is defined by $(i, 2\rho) = (i,i)$ for every $i \in I$. Notice that :

$$f(\alpha - li) - f(\alpha) + 2l(i, \alpha) = l(l-1)(i,i).$$

For any $(i,l) \in I_\infty$, since $\Omega b_{i,l} = b_{i,l} \Omega K_{2li}$, we get :

$$\begin{aligned} Cb_{i,l}m &= v^{f(\alpha - li)} \Omega b_{i,l} m \\ &= v^{f(\alpha - li)} b_{i,l} \Omega K_{2li} m \\ &= v^{f(\alpha - li) + 2l(i,\alpha)} b_{i,l} \Omega m \\ &= v^{f(\alpha - li) + 2l(i,\alpha) - f(\alpha)} b_{i,l} Cm \\ &= \begin{cases} v^{l(l-1)(i,i)} b_{i,l} Cm & \text{if } i \in I^{\text{im}} \\ b_{i,l} Cm & \text{if } i \in I^{\text{re}}. \end{cases} \end{aligned}$$

Hence, if m is a primitive vector of the kernel of the map $\oplus_{(i,l) \in I_\infty} M(-li) \to M(0)$ with $|m| = \alpha \in -\mathbb{N}I$, we have :

$$f(\alpha) = \sum_{1 \leq k \leq r} l_k(l_k - 1)(i_k, i_k) \tag{2.29}$$

where $\sum_{i \in I^{\text{im}}} \alpha_i i = \sum_{1 \leq k \leq r} l_k i_k$. Since $(\alpha, i) \geq 0$ for any real vertex i, we also have :

$$\begin{aligned} (\alpha, \alpha + 2\rho) &= \sum_{i \in I} \alpha_i (i, \alpha + i) \\ &= \sum_{i \in I^{\text{re}}} \alpha_i (i, \alpha) + 2 \sum_{i \in I^{\text{re}}} \alpha_i + \sum_{i \in I^{\text{im}}} \alpha_i (i, \alpha + i) \\ &\leq 2 \sum_{i \in I^{\text{re}}} \alpha_i + \sum_{i \in I^{\text{im}}} \alpha_i (i, \alpha + i). \end{aligned}$$

Combining with 2.29, we get :

$$\begin{aligned}\sum_{1\leq k\leq r} l_k(l_k-1)(i_k,i_k) &\leq 2\sum_{i\in I^{\mathrm{re}}}\alpha_i+\sum_{i\in I^{\mathrm{im}}}\alpha_i(i,\alpha+i)\\ &=2\sum_{i\in I^{\mathrm{re}}}\alpha_i+\sum_{i\in I^{\mathrm{im}}}\alpha_i(\alpha_i+1)(i,i)+\sum_{\substack{i\in I^{\mathrm{im}}\\ j\neq i}}\alpha_i\alpha_j(i,j)\end{aligned}$$

and thus :

$$0\leq 2\sum_{i\in I^{\mathrm{re}}}\alpha_i+\sum_{\substack{i\in I^{\mathrm{im}}\\ j\neq i}}\alpha_i\alpha_j(i,j)+\sum_{i\in I^{\mathrm{im}}}(i,i)\Big(\alpha_i(\alpha_i+1)-\sum_{i_k=i}l_k(l_k-1)\Big).$$

Since $\sum_{i_k=i} l_k=-\alpha_i$, we have :

$$\alpha_i(\alpha_i+1)-\sum_{i_k=i}l_k(l_k-1)=|\alpha_i|(|\alpha_i|-1)-\sum_{i_k=i}l_k(l_k-1)\geq 0.$$

But we also have $\alpha_i\leq 0$, $(i,j)\leq 0$ when $i\neq j$, and $(i,i)\leq 0$ when i is imaginary, hence :

$$2\sum_{i\in I^{\mathrm{re}}}\alpha_i+\sum_{\substack{i\in I^{\mathrm{im}}\\ j\neq i}}\alpha_i\alpha_j(i,j)+\sum_{i\in I^{\mathrm{im}}}(i,i)\Big(\alpha_i(\alpha_i+1)-\sum_{i_k=i}l_k(l_k-1)\Big)\leq 0.$$

Finally every term in the sum is equal to 0, and $-\alpha$ is a sum of pairwise othogonal imaginary vertices. Since the restriction of $\langle -,-\rangle$ to $\tilde{U}^-[-\mathbb{N}i]$ is nondegenerate for any imaginary vertex i, the proof is over. $\square$

Theorem 2.30. *We have an isomorphism of Hopf algebras $\Psi : U_{\mathbb{Z}}^+ \xrightarrow{\sim} \mathcal{K}$ defined by :*

$$\begin{cases} E_{i,a}\mapsto[\mathbf{1}_{ai}] & \text{if } i\in I^{\mathrm{im}}\\ E_i^{(a)}\mapsto[\mathbf{1}_{ai}] & \text{if } i\in I^{\mathrm{re}}\end{cases}$$

and mapping $\langle -,-\rangle$ to the geometric form $\{-,-\}$.

Proof. First, Ψ is defined. Indeed, we know from the Jordan quiver case that the elements $(\mathbf{1}_{ai})_{a\geq 1}$ commute if i is isotropic. Moreover the higher order Serre relations are satisfied for real vertices (see [Lus10, 7]), and, applying the Fourier transform on the imaginary vertices, we can assume that we are working with nilpotent representations. Hence we have $\mathbf{1}_{ai}=\overline{\mathbb{Q}}_{l\{0_a\}}$ as if there were no loops, and the higher order Serre relations are still satisfied. For the same reason, we know that :

$$\{\mathbf{1}_{ai},\mathbf{1}_{ai}\}\in 1+v^{-1}\mathbb{N}[\![v^{-1}]\!].$$

Hence, setting $\langle E_{i,a},E_{i,a}\rangle=\{\mathbf{1}_{ai},\mathbf{1}_{ai}\}$, $\langle -,-\rangle$ is nondegenerate (thanks to 2.12). Therefore Ψ is injective, and since Ψ is also surjective by 1.14, we get the result. $\square$

3 Relation with constructible functions

We denote by $\bar{h} : t(h) \to s(h)$ the opposite arrow of $h \in \Omega$, and $\bar{Q}$ the quiver $(I, H = \Omega \sqcup \bar{\Omega})$, where $\bar{\Omega} = \{\bar{h} \mid h \in \Omega\}$: each arrow is replaced by a pair of arrows, one in each direction, and we set $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$.

For any pair of I-graded $\mathbb{C}$-vector spaces $V = (V_i)_{i \in I}$ and $V' = (V'_i)_{i \in I}$, we set :

$$\bar{E}(V, V') = \bigoplus_{h \in H} \mathrm{Hom}(V_{s(h)}, V'_{t(h)}).$$

For any dimension vector $\alpha = (\alpha_i)_{i \in I}$, we fix an I-graded $\mathbb{C}$-vector space V_α of dimension α, and put $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$. The space $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$ is endowed with a symplectic form :

$$\omega_\alpha(x, x') = \sum_{h \in H} \mathrm{Tr}(\epsilon(h) x_h x'_{\bar{h}})$$

which is preserved by the natural action of G_α on $\bar{E}_\alpha$. The associated moment map $\mu_\alpha : \bar{E}_\alpha \to \mathfrak{g}_\alpha = \oplus_{i \in I} \mathrm{End}(V_\alpha)_i$ is given by :

$$\mu_\alpha(x) = \sum_{h \in H} \epsilon(h) x_{\bar{h}} x_h.$$

Here we have identified $\mathfrak{g}_\alpha^*$ with $\mathfrak{g}_\alpha$ via the trace pairing.

Definition 3.1. An element $x \in \bar{E}_\alpha$ is said to be *seminilpotent* if there exists an I-graded flag $\mathsf{W} = (\mathsf{W}_0 = \{0\} \subset \ldots \subset \mathsf{W}_r = V_\alpha)$ of V_α such that :

$$\begin{array}{ll} x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_{\bullet-1} & \text{if } h \in \Omega, \\ x_h(\mathsf{W}_\bullet) \subseteq \mathsf{W}_\bullet & \text{if } h \in \bar{\Omega}. \end{array}$$

We put $\Lambda(\alpha) = \{x \in \mu_\alpha^{-1}(0) \mid x \text{ seminilpotent}\}$.

The following is proved [Boz13a] :

Theorem 3.2. *The subvariety $\Lambda(\alpha)$ of $\bar{E}_\alpha$ is Lagrangian.*

Following [Lus00], we denote by $\mathcal{M}(\alpha)$ the $\mathbb{Q}$-vector space of constructible functions $\Lambda(\alpha) \to \mathbb{Q}$, which are constant on any G_α-orbit. Then, we set $\mathcal{M} = \oplus_{\alpha \geq 0} \mathcal{M}(\alpha)$ which is a graded algebra once equipped with the product $*$ defined in [Lus00, 2.1].

For $Z \in \mathrm{Irr}\,\Lambda(\alpha)$ and $f \in \mathcal{M}(\alpha)$, we put $\rho_Z(f) = c$ if $Z \cap f^{-1}(c)$ is an open dense subset of Z.

If $i \in I^{\mathrm{im}}$ and (l) denotes the trivial composition or partition of l, we denote by $1_{i,l}$ the characteristic function of the associated irreducible component $Z_{i,(l)} \in \mathrm{Irr}\,\Lambda(le_i)$ (the component of elements x such that $x_h = 0$ for all $h \in \Omega(i)$). If $i \notin I^{\mathrm{im}}$, we just denote by 1_i the function mapping to 1 the only point in $\Lambda(e_i)$.

We have $1_{i,l} \in \mathcal{M}(le_i)$ for $i \in I^{\mathrm{im}}$ and $1_i \in \mathcal{M}(e_i)$ for $i \notin I^{\mathrm{im}}$. We denote by $\mathcal{M}_\circ \subseteq \mathcal{M}$ the subalgebra generated by these functions.

The following was proved in [Boz13a] :

Proposition 3.3. *For every $Z \in \operatorname{Irr}\Lambda(\alpha)$, there exists $f \in \mathcal{M}_\circ(\alpha)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ if $Z' \neq Z$.*

Proposition 3.4. *There exists a surjective morphism $\Phi : U^+_{v=1} \to \mathcal{M}_\circ$ defined by :*

$$\begin{cases} E_{i,a} \mapsto 1_{i,l} & \text{if } i \in I^{\mathrm{im}} \\ E_i \mapsto 1_i & \text{if } i \in I^{\mathrm{re}}. \end{cases}$$

Proof. The morphism is well defined : first, the higher order Serre relations are mapped to 0. Indeed, they are for real vertices (see [Lus91, 12.11] and [Lus10, chapitre 7]), and we work with semi-nilpotent representations. Hence they are still satisfied by definition of $Z_{i,(l)} \in \operatorname{Irr}\Lambda(le_i)$ (x such that $x_h = 0$ for all $h \in \Omega(i)$). On the other hand, the commutators $[E_{i,l}, E_{i,k}]$ are also mapped to 0 if i is isotropic, thanks to the following lemma :

Lemma 3.5. *Let Q be the Jordan quiver. We set $I = \{\circ\}$ and $1_k = 1_{\circ,k}$. We have $[1_m, 1_n] = 0$ for all $m, n \in \mathbb{N}$.*

Proof. Consider $(x, y) \in \Lambda(n+m)$, and set $V = \mathbb{C}^{n+m}$. We have :

$$1_m * 1_n(x,y) = \chi\left(\left\{ W \in \mathrm{Grass}_n V \;\middle|\; \begin{array}{l} W \text{ is } (x,y)\text{-stable} \\ x^{|W}_{|W} = 0 \\ x^{|V/W}_{|V/W} = 0 \end{array} \right\}\right).$$

This is equal to 0 except if $x \in \mathcal{O}_\lambda$, where $\lambda = (\lambda_1 \geq \lambda_2)$. Then :

$$1_m * 1_n(x,y) = \chi\left(\left\{\bar{W} \in \mathrm{Grass}_{n-\lambda_2}\overline{\ker x} \mid \bar{W}\ \bar{y}\text{-stable}\right\}\right)$$

where $^-$ stands for the quotient by $\operatorname{Im} x$. Also :

$$1_n * 1_m(x,y) = \chi\left(\left\{\bar{W} \in \mathrm{Grass}_{m-\lambda_2}\overline{\ker x} \mid \bar{W}\ \bar{y}\text{-stable}\right\}\right).$$

Since $n - \lambda_2 + m - \lambda_2 = \lambda_1 - \lambda_2 = \dim \overline{\ker x}$, we get the result by duality :

$$\begin{array}{rcl} \mathrm{End}(\overline{\ker x}) & \xrightarrow{\sim} & \mathrm{End}((\overline{\ker x})^*) \\ \bar{y} & \mapsto & [\phi \mapsto \phi \circ \bar{y}]. \end{array}$$

□

Finally, the surjectivity comes from the definition of $\mathcal{M}_\circ$. □

We conjecture that Φ is an isomorphism, which should be proved by comparing the two "crystal" structures on $\mathcal{K}$ and $\mathcal{M}_\circ$ given by the following sets of bijections :

$$\mathcal{B}_{\alpha,i,\gamma} \xrightarrow{\sim} \mathcal{B}_{\alpha-\gamma i,i,0} \times \mathcal{B}_{\gamma i}$$
$$\operatorname{Irr}\Lambda(\alpha)_{i,\gamma} \xrightarrow{\sim} \operatorname{Irr}\Lambda(\alpha-\gamma i)_{i,0} \times \operatorname{Irr}\Lambda(\gamma i),$$

the latter being obtained in [Boz13a]. To that end, the notion of crystal should be generalized, and results analogous to those obtained in [KS97] should be proved.

Conclusion

Si une étape vers la dernière conjecture formulée passe par une définition généralisée des cristaux, comme proposée en fin de première partie (section 3), il faudrait d'abord étudier la catégorie $\mathcal{O}$ associée à notre groupe quantique généralisé U_v, comme entamé dans la preuve de la non-dégénérescence de la forme de Hopf associée. En particulier, l'étude des modules de Verma $M(\lambda)$, définis de manière analogue au cas usuel :

$$M(\lambda) = \frac{U_v}{\sum_{\iota \in I_\infty} U_v E_\iota + \sum_{i \in I} U_v (K_i - v^{(i,\lambda)})}$$

et des modules simles de plus haut poids $V(\lambda)$ généralement obtenus comme quotients des modules de Verma doit être menée.

Ensuite, il faudrait suivre un programme analogue à celui de Kashiwara, en commençant par définir des bases cristallines $\mathcal{B}(\lambda)$ (en oubliant abusivement le réseau associé) sur les U_v-modules $V(\lambda)$ de plus haut poids λ, ainsi qu'une base cristalline $\mathcal{B}(\infty)$ de la partie positive U_v^+, grâce à des opérateurs de Kashiwara généralisés (qui devraient être donnés par les fonctions $\delta_{i,l}$ et $\delta^{i,l}$ définis en seconde partie).

Quelques résultats techniques sur les cristaux et leurs produits tensoriels devraient alors permettre d'obtenir des résultats généralisant ceux décrits en introduction, qui, couplés aux résultats géométriques de cette thèse, pourraient ensuite avoir plusieurs applications, notamment la définition d'une base semi-canonique. Mieux, on devrait obtenir des isomorphismes de cristaux :

$$\mathcal{B}_{\text{Perv}} \simeq \mathcal{B}(\infty) \simeq \operatorname{Irr} \Lambda$$

où $\mathcal{B}_{\text{Perv}}$ désigne le cristal des isoclasses de faisceaux pervers simples de Lusztig.

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